chalkdust

ISSUE 02 // AUTUMN 2015

JOY OF JACOBIANS

FIELDS MEDALLIST ARTUR AVILA

FRACTOGRAMS

NASH'S LEGACY

THE PERILS OF P-VALUES

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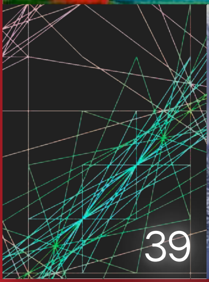

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A lot of fantastic things happened after the rush and excitement of our first issue: we started a weekly blog post, we distributed a monthly newsletter and we continued to build the foundations for a successful project. We wanted to provide a space for mathematicians and for those who like *mathsy* stuff to share their own ideas and so we were filled with enthusiasm when we began to receive articles and material for this second issue, including a wonderful cartoon. Also, and to our surprise, we were able to interview Artur Avila, a 2014 Fields Medallist and a celebrity in the mathematical world. For us it was an experience to remember and a story that is well worth sharing.

At the start, we did not know how our maths magazine would be received and we wondered whether anyone would bother reading it. But we have been astonished by the number of readers that have flipped through the pages of our first issue, whether it be the printed or online version; by the number of people who submitted an answer to our crossnumber; by the regular followers who, week after week, look at our posts; and by the likes, the shares and the tweets we have received.

And none of this would ever have happened if it weren't for you, our readers. Every click on our website, every like and every tweet motivates us to carry on, so thank you for your encouragement: maths is everywhere; it's fun, entertaining, useful and beautiful and we are proud to be sharing some of the sparks and magic it has to offer.

Rafael Prieto Curiel
Editorial Director

Acknowledgements

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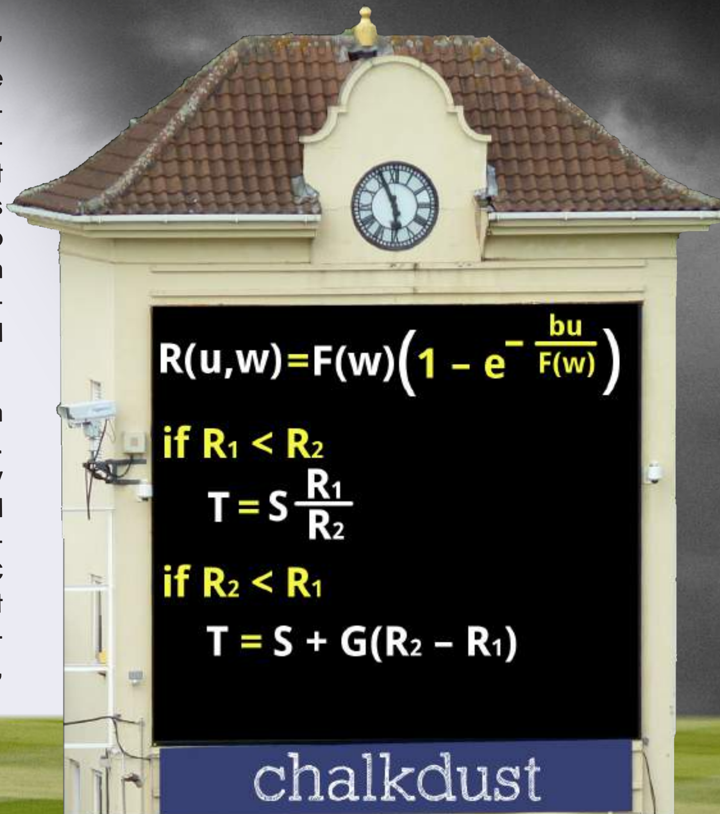
page 3 model

If there are two things that typify an English summer, they are cricket and rainy days. Unfortunately, the two very often come together, which makes it very difficult to decide who should win a limited overs cricket match when rain stops play.

In these cases, a statistical model known as the *Duckworth-Lewis method*, devised by statistician Frank Duckworth and mathematician Tony Lewis, settles the issue (and provokes copious debate amongst Lord's Long Room members as they sip their champagne).

The arrival of rain could reduce the teams' batting resources—a combination of the balls left to bowl (u) and the number of players not out ($10 - w$). A team's remaining resources can be modelled by the first equation on the scoreboard, where $F(w)$ is the proportion of runs you would expect to score with w wickets lost compared to with no wickets lost; and b is an exponential decay constant. These are both calculated from historical cricket data.



If the rain stops, the target score, T, can be updated using the other two formulae. Here S is the number of runs obtained by Team 1; G is the average score expected from the team batting first in an uninterrupted match, published annually in the ICC Playing Handbook; and R_1 and R_2 represent the resource percentage relative to a full innings available to each team respectively, calculated using the first formula.

Now where's that champagne?

In conversation with... Artur Avila

Anna Lambert & Rafael Prieto Curiel

SITTING in a pub in Leicester Square, talking to one of the most brilliant mathematicians of our generation, is not the way one would normally expect to spend a sultry evening in early June. But it turns out that Artur Avila, winner of the 2014 Fields Medal, takes very spontaneous holidays, and is a big fan of the pub.

Avila certainly does not conform to the UK's stereotype of a mathematician. He is good-looking, stylishly dressed in a white T-shirt and designer jeans and asking about the best London night-clubs. However, Avila was born and bred in Rio de Janeiro, famous for its spectacular parties and beautiful beaches. He still spends half his time there, based at the National Institute of Pure and Applied Mathematics (IMPA), and spends the other half at the French National Centre for Scientific Research (CNRS) in Paris where, according to Avila, the nightlife is terrible.

It's perhaps unusual for such an exceptional mathematician not to spend all their time in the USA or Europe, where there are higher numbers of world class research institutions but Avila believes "it's significant that I studied at IMPA because it shows that Brazil has institutions that can prepare someone to do maths at a high level, and it's not necessarily true that you always have to go to the United States or to Europe to advance".

And, of course, Rio has many appealing features: "I have several times brought collaborators to the beach with me and we would just sit and share ideas with each other, with the sound of the sea in the background."

There are clearly advantages of this arrangement to Brazil too. In 2014, Avila won the Fields Medal—a prize often referred to as the equivalent of the Nobel Prize for mathematics, awarded to up to four mathematicians every four years. Whilst the Nobel Prize and Fields Medal may be similar in terms of prestige attached to them, in reality they have very different aims. The Nobel Prize is often awarded years after a discovery has been made, when its full impact has become evident. The Fields Medal on the other hand is only awarded to mathematicians under 40 years of age, in the belief that this will stimulate further work. Often, mathematicians are awarded the prize on their "last chance"; that is, in the last awarding year before they are 40. This was true of all four winners in 2010 and the other three winners in 2014. However Avila was just 35, and the work that won him the medal had been completed years earlier.

"We would just sit and share ideas with each other, with the sound of the sea in the background."

The Fields Medal announcements in 2014 received particular attention because it was the first time in its 78 year history that it was awarded to a woman, Maryam Mirzakhani. Less publicised was the fact that it was also the first time a Latin American has won a Fields Medal. Indeed, no Brazilian scientist has ever won a Nobel Prize either, and Avila believes that "there is a tradition in Brazil to think that no science that comes from there has any quality, and it's good to give examples that that's not the case". Indeed, he is convinced that the saturated nature of the academic job market in the West and the constant battle for research positions and funding will make universities in ambitious, rapidly developing countries such as Brazil much more attractive. Which, "once you have the type of people who have good plans and lots of energy, and who do high quality work", will further help to make the world sit up and take notice.

Strangely enough, the 2014 Fields Medal was not the first time that Avila and Mirzakhani have shared success. They both won gold medals at the International Mathematical Olympiad (IMO) in 1995 in Toronto and Avila recalls that "she sat next to me during the prize ceremony, and then of course again in Seoul in 2014."

"And of course, the IMO makes you realise that maths can be fun."

The IMO is an annual mathematics competition for pre-university students, of extreme difficulty. A quarter of all Fields Medallists have won medals at the IMO, which is even more incredible given that it began over twenty years after the Fields Medal was first awarded. Countries have rigorous training regimes, typically consisting of intensive camps where students work on problems similar to those that they will face in the competition. Like many others, Avila believes that training for and competing in the IMO was a major formative experience in the process that led him to become a mathematician: "It was the first structured thing that came along that was really challenging. When you start, you struggle with these unfamiliar problems, but that gives you focus to work very hard on them. And of course, it makes you realise that maths can be fun."

The training was held at IMPA, which opened up the world of mathematics research to Avila. Although he enjoyed the Olympiad, he decided not to participate in further competitions after returning with the gold medal from Toronto at the age of 16, but to immediately begin his undergraduate education at IMPA. At 21, he completed his doctorate, having already proved some

George Ioannidis

The trajectory of a chaotic double pendulum

outstanding results in his chosen area of dynamical systems.

Dynamical systems is a branch of mathematics that considers how a point in space moves over time if you apply a fixed rule to it repeatedly. A classic example of this is the motion of the bob of a pendulum. The central aim is to determine the behaviour of the point after a long period of time, and there are three main possibilities. The first is periodic behaviour, such as the usual back and forth oscillations of a pendulum. However, if the air resistance is drastically increased, the pendulum would eventually come to a halt. In dynamical systems, the system reaches a fixed point. Both fixed points and periodic behaviour are fairly intuitive, but the third possibility—chaos—is much more surprising.

"If they were not that hard, then somebody would have solved them already."

If a second pendulum is attached to the end of the first pendulum, its behaviour becomes very complex. Sometimes the bottom pendulum may flip over completely rather than oscillating from side to side. Whether this flip happens, and when it happens, depends on the position from which the pendulum is first released. This dependence on the initial conditions is extremely sensitive and very counterintuitive: raising the bottom pendulum ever so slightly may cause it to take 1,000 times longer to flip over. This behaviour is known as chaos, and it leads to trajectories that look completely irregular, each entirely different to another where the pendulum has been released from a slightly different position.

Although the behaviour is entirely deterministic, in practice it is so complicated that it is best understood in a probabilistic manner. In Avila's words, "the interesting part is that you start with a deterministic system that apparently doesn't have any randomness inside it, and due to the complexity of the system, it is better modelled by something that is very random."

"Communicating is a bit difficult because ideally you don't want to lie. The mathematical reality is complicated."

Understanding which systems display chaos was a huge strand of research in the 1970s. However, several important questions remained open. One of these concerned a general class of dynamical systems known as unimodal maps. Mathematicians believed that these could be categorised according to their long term behaviour as either regular or stochastic. Regular systems eventually show periodic behaviour or converge towards a fixed point, but stochastic systems have chaotic orbits that appear random, and hence are best understood using probabilistic tools. In 2003, Avila and his collaborators showed this to be true, proving that a randomly chosen unimodal map will either be regular or stochastic. This exceptional result provided an overarching understanding of these systems, and was the culmination of a long line of research.

This is only one of Avila's groundbreaking results, but he is generally unwilling to reduce the details of his work to a neat, easily understood analogy. "Communicating is a bit difficult because ideally you don't want to lie. The mathematical reality is complicated." For the general public, he believes that the increased visibility of mathematics and mathematicians after the announcement of his Fields Medal is more important than the exact specifics of his work. The hope is that some young people may be inspired to follow more in his footsteps than those of Messi!

When talking to Avila about his research, he projects a certain sense of ease. He is driven to do mathematics because he is "just kind of curious" and he tries to satisfy that curiosity wherever he is. Astonishingly, he often does mathematics without writing anything down, and has made breakthroughs on the train into work and on flights from Rio to Paris. "The advantage of making the computation without paper is that your memory restricts the complexity of the problem, so you have to structure the question in a smarter way. When you finally succeed you have a better understanding than if you used brute force."

Artur Avila poses with Issue 1 of *Chalkdust*.

This is not the only way in which Avila seems relaxed and clear-headed about his research. Stories of mathematicians dedicating years to solving one problem, and the emotional highs and lows that come with it, are commonplace. That's not for Avila. "I don't find it very smart to fix your ideas on one famous problem. They are often essentially technical challenges that are extremely hard, because if they were not that hard, then somebody would have solved them already."

Instead, he prefers to work on many different problems and dive into areas that he knows little about. "I come as an outsider and look at the traditional problems in the field, but without knowing the usual methods that people use. In the past I have made conjectures that were initially completely wrong because I was so unfamiliar with the topic. But in the end the new approach solved the problem." This wide ranging approach has been very fruitful, and Avila has worked with over 30 collaborators worldwide.

These collaborations show the more social side to Avila, and bring us back to the pub. Conversation over, he finishes the rest of his whisky and sets off towards the bright lights of Leicester Square and its clubs, ready to party the night away.

Anna Lambert is a PhD student at UCL working on mathematical models of bioreactors. You can contact her on Twitter ***@anna_lambert****.*

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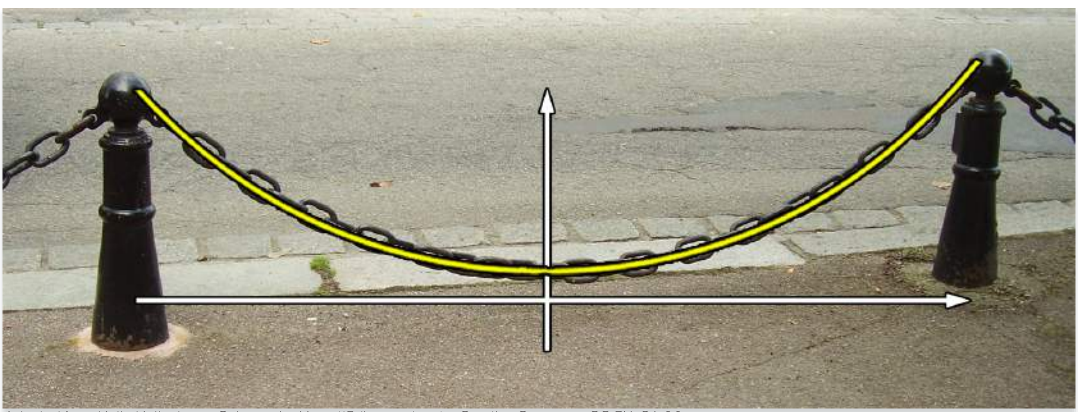
My Favourite Function

Functions are rather important. From the humble y = constant to the exotic special functions, from straight lines to never-ending spirals, from those in one variable to those that live in multidimensional worlds, they're pretty hard to escape. We have spread some fascinating functions throughout this issue. We'd really love to hear about your favourites! Send them to us at **contact@chalkdustmagazine.com**, on Twitter **@chalkdustmag** or at **facebook.com/chalkdustmag**, and you might just see them on our blog!

Hyperbolic Cosine

Adam Townsend

How many functions do you see drawn every day? This definition of this function looks initially a bit odd (even though it's even, ha ha), $\cosh(x) = (e^x + e^{-x})/2$, and you might wonder why it's named after a trig function. But if you draw it, you get the same shape as when you hang a chain between two points! This shape is called a catenary (*ka-TEEN-uh-ree*), and the fact that this function turns up in other parts of maths as well (you first see it when solving certain differential equations), I think is really cool.



what's hot and what's not

Python

```
from jokes import jokes_about_python
print(jokes_about_python)
```

Fortran

We don't use punch cards any more. Why are you still using a language designed for them?!

Isaac Newton

Over 300 years before it was cool, Newton invented the diss track and completely destroyed Leibniz.

Gottfried Leibniz

His haircut is so last issue and he doesn't even have an Instagram account. Not cool.

Choco Leibniz

Yum!

Fig Newtons

Yuck!

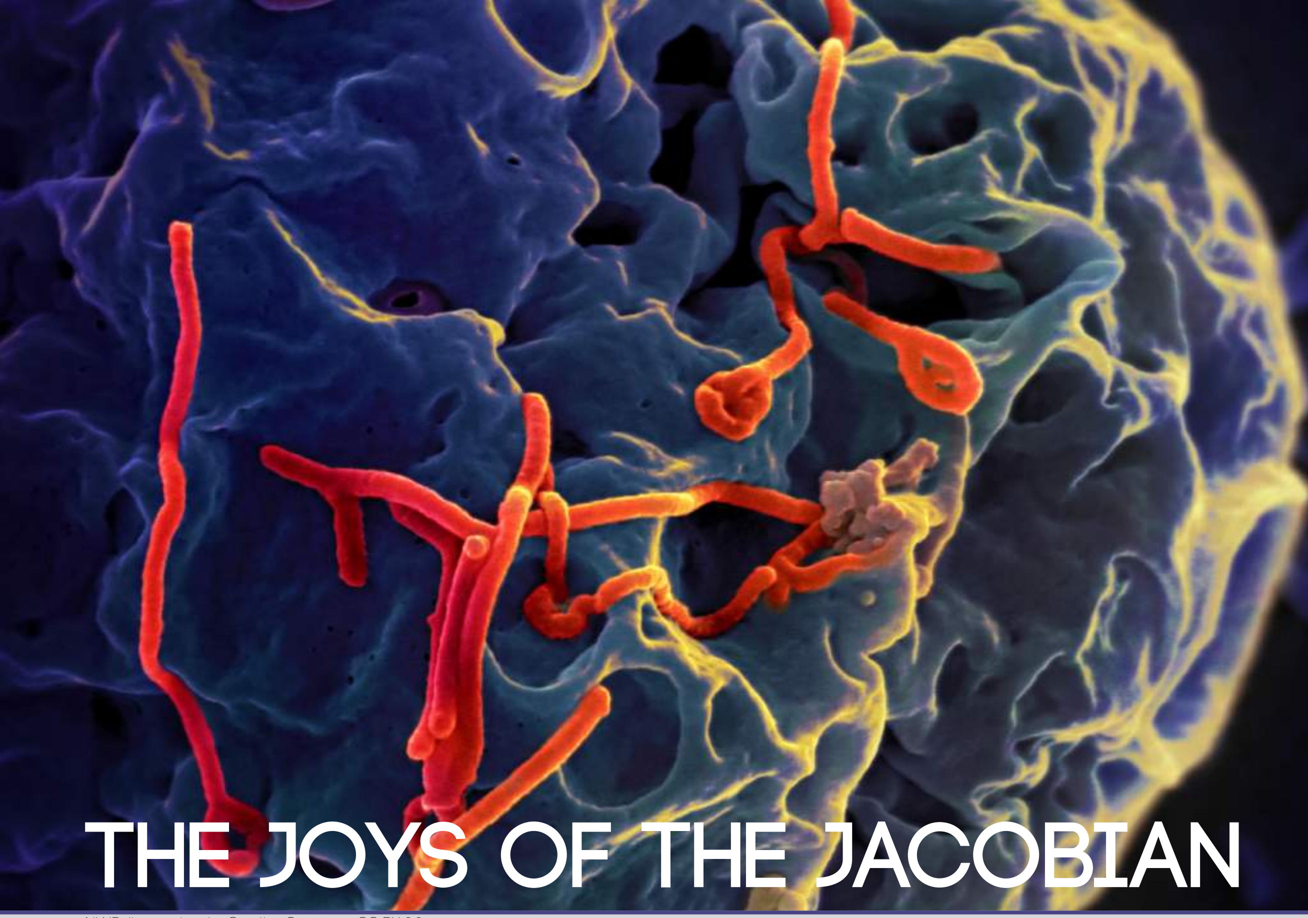
36.7°C

The UK's hottest ever July temperature.

36.7°F

Not the UK's hottest ever July temperature.

Agree? Disagree? Let us know on Twitter **@chalkdustmag**, at **facebook.com/chalkdustmag** or at **contact@chalkdustmagazine.com.**



THE JOYS OF THE JACOBIAN

Robert Smith?

I am an applied mathematician. Like, *very* applied. I use mathematics, sure, but I'm far more interested in what maths can do to solve real problems. Specifically, how can mathematics be used to tackle issues in infectious diseases?

The best thing about maths, from my point of view, is that it has one incredible superpower: it can predict the future. That's amazing. And very useful, of course.

> *The best thing about maths, from my point of view, is that it has one incredible superpower: it can predict the future.*

When teaching my undergraduate students the required details to make these predictions, I stumbled upon a very profound realisation. Namely, that the Jacobian matrix—a technical thing from linear algebra—is a) just about the most massively useful thing ever and b) a glorious way to reconcile two apparently disparate strands of mathematics.

Don't believe me? Read on and hopefully you too will become a convert to the church of the Jacobian...

Eigenvalues: why?

Being an applied mathematician, the issue of eigenvalues and eigenvectors always puzzled me. Specifically, why on earth would you want to deal with them in the first place? I'm all for doing maths for its own sake, but I often have to explain mathematics to people who a) aren't mathematicians and b) live their life in sheer terror of equations.

(No, since you ask, they're not all politicians. Okay, some of them are politicians. But not all. I work with biologists, epidemiologists and policy-makers, who need to use maths to make predictions, but would often be very happy if they never saw a Greek letter ever again.)

So to explain dizzying mathematical concepts to such people, you need to do it in a very grounded way. That is, absolutely everything needs a rock-solid foundation. Usually that's not a problem, because the types of maths that they'll encounter (disease models using differential equations) all have a very sensible grounding in the real world that can give them a handle on it. But not eigenvalues. That's a tough one.

The best explanation I could come up with was this: If $\mathbf{A}$ is a matrix and $\mathbf{x}$ is a vector, then $\mathbf{Ax}$ is also a vector. So could it be possible that $\mathbf{Ax} = \mathbf{x}$ somehow? (Okay, this isn't exactly "the real world" at its finest, but bear with me.)

First off, if this is the case, then that tells us something about the dimension of $\mathbf{A}$: it has to be a square matrix. That is, there must be as many columns in $\mathbf{A}$ as there are rows in $\mathbf{x}$ (or else we wouldn't be able to multiply them together). But if $\mathbf{Ax} = \mathbf{x}$, then there must also be as many rows in $\mathbf{A}$ as there are in $\mathbf{x}$ (or else the product would produce a vector of a different size). That's pretty neat, actually.

Centers for Disease Control and Prevention

Malaria, carried by mosquitoes, kills a child every minute in Africa. The Jacobian helps mathematicians model how effective countermeasures could be.

Second, my non-mathematician friends can solve this themselves, because the answer is clearly $\mathbf{x} = \mathbf{0}$. Hooray, that was easy! Except... well duh, of course it is. So the real question is: are there any vectors *other* than zero? Sorry non-maths buddies, we're not letting you off the hook that easily.

Third, let's be slightly more general. Let's suppose that multiplying $\mathbf{A}$ by $\mathbf{x}$ doesn't just equal $\mathbf{x}$, but can equal some scalar multiple of $\mathbf{x}$ (which we'll call λ). This gives us the core equation

$$\mathbf{Ax} = \lambda\mathbf{x}. \tag{1}$$

At this point, if you're a non-mathematician, you'll probably throw your hands up and storm off. Not because you've seen an equation (heaven forbid), but because you probably know just enough to know that there's a fundamental issue here and it's this: *this is a ridiculous problem.*

No, really, it is. If $\mathbf{A}$ is known and I can safely say that $\mathbf{A}$ is an $n \times n$ matrix, then solving $\mathbf{Ax} = \mathbf{x}$ is bad enough, because you have to solve n equations for n unknowns. Okay, fine. But if we add in the issue of λ, then you have at least $n + 1$ unknowns... and that's if there's only one λ. (Spoiler:

there isn't.) In fact, there's now a second problem, which is: how do we even know how many λs will satisfy this equation? Those crazy mathematicians...

And, I have to say, that's an excellent question. The problem appears far too big to solve in any sensible way. But let's proceed nonetheless. (Yep, we mathematicians are kind of crazy. That part isn't in doubt.) If these were numbers, I'd just move everything over to one side and divide. I can't *quite* do that with matrices and vectors, but let's see how we go anyway:

$$
\begin{aligned}
\mathbf{Ax} &= \lambda\mathbf{x} \\
\mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} && \text{(subtracting the right-hand side)} \\
(\mathbf{A} - \lambda)\mathbf{x} &= \mathbf{0} && \text{(factoring out the } \mathbf{x}\text{).}
\end{aligned}
$$

Bzzzzt! Uh-uh, you can't do that. Why not? Because $\mathbf{A}$ is a matrix and λ is a number, so they can't be subtracted from one another. (You can multiply a matrix by a number, sure, but you can only add or subtract things of the same dimension.) This is where the fact that we're dealing with matrices rather than numbers starts to throw up issues.

Fortunately, we have a secret weapon at our disposal and—you guessed it—it's the identity matrix. (Okay, maybe you didn't guess it. That's okay, that's why it's a secret.) The identity matrix is essentially the matrix equivalent of the number 1. If you multiply something by it, that something is unchanged. It also happens to be square.

In particular, $\mathbf{Ix} = \mathbf{x}$. Which... okay, sure, is this going somewhere? ask my non-mathematician friends, whose patience is surely being tried at this point. But the answer is yes, because of the fact that equals signs work in both directions.

Usually we'd think that we're multiplying $\mathbf{x}$ by $\mathbf{I}$. Which isn't terribly interesting. But because equals signs work both ways, we can do the reverse too: we can take $\mathbf{x}$ and *insert* the identity matrix $\mathbf{I}$ in front of it. Why would we want to do that? Because it solves our "λ" problem! Let's see:

$$
\begin{aligned}
\mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} \\
\mathbf{Ax} - \lambda\mathbf{Ix} &= \mathbf{0} && \text{(inserting } \mathbf{I}\text{)} \\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} && \text{(factoring out the } \mathbf{x}\text{).}
\end{aligned}
$$

See, there's no problem now. $\mathbf{A}$ is a square matrix (it had to be, remember) and $\mathbf{I}$ is also a square matrix, so things line up perfectly.

If these were numbers, I'd just divide... so long as I knew I wasn't dividing by zero. With matrices, we do something similar: we take the inverse... so long as the matrix is actually invertible.

Actually, I have no idea whether the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is invertible or not. How would I? I don't have the first clue what λ is. So let's consider both possibilities.

Possibility 1: $(\mathbf{A} - \lambda\mathbf{I})$ is invertible. Right, off we go.

$$\begin{aligned}(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0}\\(\mathbf{A} - \lambda\mathbf{I})^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0} \qquad \text{(apply the inverse to both sides, on the left)}\\\mathbf{x} &= \mathbf{0}.\end{aligned}$$

D'oh. My non-maths friends could have told you that. In fact, they did. So if $(\mathbf{A} - \lambda\mathbf{I})$ is invertible, then there's only the trivial solution and we explicitly excluded that. So that means that $(\mathbf{A} - \lambda\mathbf{I})$ cannot be invertible.

Possibility 2: $(\mathbf{A} - \lambda\mathbf{I})$ is not invertible. There's another way to say this:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{2}$$

Woah. Woah! Did we just figure out λ independently of $\mathbf{x}$? Hells yes! In fact, I'll go one better: since $\mathbf{A}$ is an $n \times n$ matrix, this equation is an nth order polynomial in λ. No, wait, I can do even better than that: the fundamental theorem of algebra states that an nth order polynomial not only has a solution, it has precisely n solutions in the complex field. (In fact, this is why we want complex numbers in the first place: to solve polynomials.)

This is magnificent. It means a) I know how many λs there are; b) I can solve equation (2) for these special values (aka "eigenvalues") first; and c) for each eigenvalue, I can use equation (1) to find the special vectors (aka "eigenvectors") $\mathbf{x}$.

Because of the noninvertibility of $(\mathbf{A}-\lambda\mathbf{I})$, I'll get an infinite number of eigenvectors corresponding to each eigenvalue, but that's okay. I solved my maths problem.

Except, I still haven't solved my biological problem. And my non-maths friends are being very patient here. Why would I want to do this in the first place?

How the Jacobian helps disease modelling

When I use differential equations to create a mathematical model of a disease (e.g. HIV, Ebola, zombies), I might write something like this:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \mu S - \beta SI\\\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI - \mu I - \gamma I.\end{aligned}$$

This is a classic disease model, called the "SI" model. It stands for Susceptible–Infected. To create such a model, you figure out what comes in and what goes out. Here, I have susceptible individuals, who can come in only by being born, at a rate Λ. Two things can happen to them: they can die (from things unrelated to the disease) or they can become infected. So they leave (for good) at rate $-\mu S$ and they transfer from the susceptible compartment to the infected compartment at rate $-\beta SI$.

Likewise, the infected individuals can only come into being by becoming infected, at rate βSI. They leave either by dying (from things unrelated to the disease), at rate $-\mu I$, or by dying from the disease itself, at rate $-\gamma I$.

Spread of Ebola in Guinea, Sierra Leone and Liberia, August 2014

Because this is a nonlinear system of differential equations, I have no hope of solving it. But I'd still like to ask crucial questions, like "What happens eventually?". Specifically, will the disease die out or will it become endemic? If the disease is the flu, this might be useful information to have. If the disease is zombies or Ebola, this might be *really* useful information to have.

In general, I can't solve differential equations. But I can solve equations! So the first step is to look at equilibria. That is, values when both derivatives are zero:

$$0 = \Lambda - \mu S - \beta SI$$
$$0 = \beta SI - \mu I - \gamma I.$$

These are two simultaneous equations with two unknowns. Easy peasy. I can solve this to find two equilibria:

$$(S, I) = \left(\frac{\Lambda}{\mu}, 0\right), \left(\frac{\mu+\gamma}{\beta}, \frac{\Lambda}{\mu+\gamma} - \frac{\mu}{\beta}\right).$$

The first one, called the disease-free equilibrium, always exists; the second, known as the endemic equilibrium, only exists some of the time. Pipe down please, mathematicians: these are infected individuals we're talking about—how can they be negative? Or, to put it another way, we'd like to avoid negative people.

Finding the endemic equilibrium can be tough as models get more complicated. But the disease-free equilibrium always exists (in any sensible model) and is easy to find because it comes with the additional constraint that $I = 0$.

What I'd really like to know is the stability of the disease-free equilibrium. Why is this so crucial? If I start on the equilibrium, I stay there of course. But suppose I start close to my equilibrium (e.g. a few infected individuals). If the equilibrium is stable, the disease will die off. If the equilibrium is unstable, then the disease will persist.

Stability of equilibria in one dimension is equivalent to looking at the slope of the tangent line at the equilibrium. In higher dimensions, we can generalise: we linearise around our equilibrium and find the analogue of the "slope". In one dimension, we simply use the derivative. But in higher dimensions, there's no such thing as "the" derivative; instead, we have many partial derivatives.

The way to make this work is to create a matrix of partial derivatives, called—you guessed it!—the Jacobian. Essentially, it's the analogue of "the" derivative in higher dimensions. You create it by differentiating every equation with respect to every variable. Thus

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}.$$

It turns out that this is just about the most massively useful thing ever. Okay, my non-maths friends are arching their eyebrows at this point, because it sure doesn't look like it from where they're sitting. But let's work through our example:

$$\mathbf{J} = \begin{bmatrix} -\mu - \beta I & -\beta S \\ \beta I & \beta S - \mu - \gamma \end{bmatrix} \qquad \text{(the first column is the derivatives with respect to } S\text{, the second with respect to } I\text{)}$$

$$\mathbf{J}\Big|_{(S,I)=\left(\frac{\Lambda}{\mu},0\right)} = \begin{bmatrix} -\mu & -\beta\Lambda/\mu \\ 0 & \beta\Lambda/\mu - \mu - \gamma \end{bmatrix} \qquad \text{(substituting the disease-free equilibrium).}$$

Happily, this matrix is upper triangular, so the eigenvalues lie on the diagonal (not true in general, of course). Our eigenvalues are thus $\lambda = -\mu$, and $\frac{\beta\Lambda}{\mu} - \mu - \gamma$. Hooray!

For any two-dimensional linear system (which ours isn't, of course), solutions look like

$$\begin{pmatrix} S \\ I \end{pmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \tag{3}$$

where c_1 and c_2 are arbitrary constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues λ_1 and λ_2, respectively. That is, we take linear combinations of two fundamental solutions. The eigenvector part of these solutions gives us the dimensionality. But the eigenvalue part is exponential, which turns out to be hugely important.

How do we know that solutions are always of the form (3)? Because we know everything there is to know about linear systems. For nonlinear systems, we only know some special cases, but linear systems have been completely solved.

Finding an eradication threshold

So let's recap. We can fully solve any linear system and the solution depends on the eigenvalues. Actually, better than that, it only depends on the sign of the eigenvalues. Actually, even better: it only depends on the sign of the real part of the eigenvalues. Why? Because these are exponentials. So if $\lambda = a + \mathrm{i}b$, then solutions are of the order

$$e^{\lambda t} = e^{(a+\mathrm{i}b)t} = e^{at}(\cos bt + \mathrm{i}\sin bt)$$

using Euler's formula. If $a > 0$, solutions increase without bound. If $a < 0$, solutions decrease to zero.

Thus linear solutions either blow up or go to zero at a rate of e^{at}. That is, if we have complex eigenvalues, we can ignore the imaginary part, because the imaginary part only contributes oscillations: i.e. it's bounded, so plays no part in the stability. If there's more than one eigenvalue, then we need all the eigenvalues to have negative real part for stability; if just one eigenvalue has positive real part, then solutions blow up. This makes sense, because while the others are happily going to zero (exponentially fast), if one is heading to infinity, then the solution overall is going to be unstable.

(If you're paying attention, you'll notice that I've left something out. What did I miss? Before you read on, try to think about the case I didn't mention.)

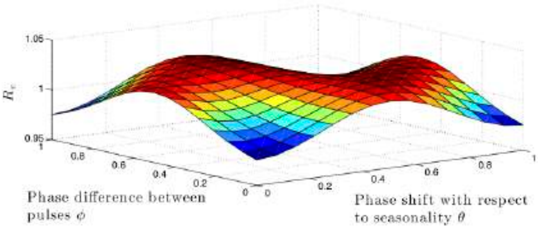
But that's for linear systems, whereas we're usually dealing with nonlinear systems. So here's the rub: if my initial conditions are sufficiently close to my equilibrium, then—in almost all cases—the behaviour in the nonlinear system can be approximated by the behaviour in the linear system. This is because a function looks a lot like its tangent, if you're sufficiently close to the point in question. That's not just true of one-dimensional functions, it's true in higher dimensions too.

So the same result applies: we take our nonlinear system, find equilibria, calculate the Jacobian and find the eigenvalues. If the real part of *any* eigenvalue is positive, then solutions blow up. If the real part of all the eigenvalues is negative, then solutions converge (locally) to the equilibrium. The case I missed? If the real part is zero. In that case, we have to use more complicated methods. But those cases are a) few in number and b) irrelevant if we take a broader view.

Looking at our example, we see that our eigenvalues are $-\mu$ and $\frac{\beta\Lambda}{\mu} - \mu - \gamma$. The first one is always negative, so it plays no part in the stability (it's not enough to prove stability, but it doesn't rule it out either). But the second one could be either positive or negative, depending. This gives us a threshold criterion. Define

$$R_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma)}.$$

If $R_0 < 1$, the disease dies out (because all the eigenvalues are negative), whereas if $R_0 > 1$, the disease persists (because there's a positive eigenvalue). This of course depends on the parameters of the disease, like the transmission rate β and the death (or recovery) rate γ, as well as characteristics of the population like the birth rate Λ and the background death rate μ.



Robert Smith?

Using R_0 to predict the eradication of polio, depending on pulse vaccination and seasonal fluctuations.

Why did I put that as a fraction instead of just talking about the eigenvalues? Two reasons: 1) because R_0 actually represents the average number of secondary infections caused by a single infected individual; and 2) because my non-maths friends don't know what an eigenvalue is, but they do know what R_0 is. It's a well-established value in the biological sciences and is one of the things we have in common with them.

So there you go. I can now fully predict the progress of my disease. I simply calculate R_0 (which I can get by finding the eigenvalues, which I get from the Jacobian matrix) and then I know everything I need to know. Even better, if I can intervene in some way, like by lowering the transmission rate β (such as through a vaccine or using condoms or changing patterns of behaviour), then I lower my R_0. If my intervention lowers R_0 sufficiently, then it will fall below 1 and then my disease will no longer persist and will instead be eradicated.

What this means is that I have a very powerful method for telling the future. I can assess my intervention methods in advance and know whether they'll be effective or not. This can save me a great deal of money and, potentially, very many lives as well.

And it all comes about because of a little thing called the Jacobian matrix. A matrix you build from a mathematical model using calculus, that you analyse using algebra and which, as we've now seen, turns out to be just about the most massively useful thing in the history of ever.

Actually, I'm not joking about that last point. It's been estimated that malaria has killed half of all humans who ever lived. Half. One in two humans who ever lived died of malaria. And yet, today, most of us in the developed world don't die of malaria. Why not? Because disease modellers used R_0 to show that spraying insecticide would switch the system from persistence to eradication. So they sprayed the world with DDT. Hence there's a very good chance that most of us are alive today, thanks to this. And thanks to the Jacobian.

So don't underestimate the power of mathematics. It might just save your life. And probably has.

*Robert Smith? (**rsmith43@uottawa.ca**) is a professor of biomathematics at the University of Ottawa. He once combined his day job with pop culture and in doing so accidentally invented the academic sub-discipline of mathematical modelling of zombies. He uses mathematical models to predict the spread of diseases, from HIV to malaria to Ebola. He's also the foremost authority on the spread of Bieber Fever, but let's not worry about that one. He has eleven books on academia and/or pop culture to his name, most recently* Mathematical Modelling of Zombies *and* The Doctors Are In: The Essential and Unofficial Guide to TV's Greatest Time Lord.

Seven Digits

I'm thinking of a number. I've squared it. I've squared the square. And I've multiplied the second square by the original number.
So I now have a number of seven digits whose final digit is a seven.
What was my original number?

Source: My Best Puzzles in Mathematics by Hubert Phillips
Answers at chalkdustmagazine.com/answers

dear dirichlet

Moonlighting agony uncle Professor Dirichlet answers your personal problems. Want the Prof's help? Send your problems to **deardirichlet@chalkdustmagazine.com**.

Dear Dirichlet,

My wife and I are having difficulty with her shift times as a Northern line tube driver. We're always tired when we see each other and I just feel that every point in our relationship ends up leading to an argument. Can you help?

— Complexified, High Barnet

DIRICHLET SAYS

According to the theory of Monsieur Argand, a quick look at the complex plane will tell you that every point has an argument. The good news is that it will also have a finite length. You may wish to coincide your disagreements with engineering work on the tube: any branch cuts will let you arrive at the same point by different arguments.

Dear Dirichlet,

My parents were away for the weekend and so I invited a handful of my best friends over for a secret party. Horrifically, hundreds of randomers from Facebook turned up and made it a complete disaster. Now my parents are furious and I'm terrified.

What should I do?

— Irrational, Reading

DIRICHLET SAYS

It sounds like you hosted a non-discrete function. This is not surprising because all the functions you see in school tend to be continuous (I'm looking at you, sin(x)). The easiest way to make your functions discrete in the future is just to limit the domain. Maybe only the integers next time?

Dear Dirichlet,

I'm taking Spanish classes to be able to speak to my boyfriend in his native language.

But when I try to speak to him at home with my new skills, he doesn't seem to appreciate it.

Help me Dirichlet!

— Infinitely Differentiable Operator, Anglesey

DIRICHLET SAYS

You appear to be experiencing translation invariance. That is to say, $\forall x$, $f(a+x)=f(a)$. Don't worry though, because there are a few things with this property that you can use even with this handicap. The less-than property is translation invariant, as is the Fourier transform. Let me know how you get on.

PS. Are you sure your boyfriend is Spanish?

Dear Dirichlet,

My supervisor gave me a right rollocking recently for citing his full name in a paper I submitted.

Apparently he feels it is essential to not only abbreviate his first names by single letters, but also to dot and space them in a certain way, and is quite touchy about it.

Is this normal?

— S.H. Moschen, Liverpool

DIRICHLET SAYS

This sounds like a classic case of sensitive dependence on initial conditions. It is quite common in professors looking to evoke the grand age of academia, when there were few enough people in a field that everyone knew each other. It is also sometimes found in PhD students thinking they are important enough to get away with it. The style guide of the paper will ultimately dictate which convention to use, so don't worry!

The Mpemba paradox

why warm water freezes faster

Image by Schnobby, licensed under Creative Commons CC BY-SA 3.0

Oliver Southwick

If you take two identical cups, fill one with warm water and one with cold and put them in the freezer, you'd expect the cooler one to freeze first, but it doesn't always. In fact, in many circumstances, it is the warm water that freezes first.

This is the Mpemba paradox, named after Erasto Mpemba, who observed it as a schoolboy in Tanzania in the 1960s. When physicist Dr Denis G. Osbourne visited his school, Mpemba took the opportunity to ask about his strange observation. Although initially skeptical, Osbourne later reproduced the observations and several years later, in 1969, they jointly published the result. Since then, it has been reproduced in many experimental studies and its origins have been debated extensively.

Mpemba wasn't the first to observe this though. The effect has been discovered and rediscovered many times over at least two millennia and caught the attention of both the British polymath Francis Bacon and, independently, his contemporary, the French mathematician and philosopher René Descartes. The earliest known reference is from Aristotle, who wrote that "the fact that the water has previously been warmed contributes to its freezing quickly: for so it cools sooner. Hence many people, when they want to cool water quickly, begin by putting it in the sun." He thought it supported his idea of antiperistasis: that a quantity is intensified by being surrounded by its opposite.

"Many people, when they want to cool water quickly, begin by putting it in the sun."

Surprisingly, there is still no scientific consensus on the exact cause of the Mpemba effect. Many people have claimed that their explanation is the definitive answer but the root of the Mpemba effect is still regarded as an open problem. It's a very interesting problem as well because it crosses disciplinary boundaries. Some explanations focus on chemistry, others look at physics. As with many problems, a clear route to the crux of this paradox is to examine it mathematically.

A mathematical view

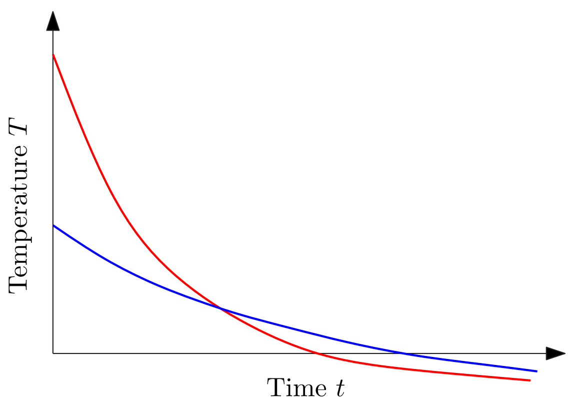
We're interested in the temperature T over time t. To find this we need to know the rate of change of temperature, $\mathrm{d}T/\mathrm{d}t$. A reasonable first model would be to say that this rate of change depends on the temperature, so we could write

$$\frac{\mathrm{d}T}{\mathrm{d}t} = f(T),$$

for some function f. But if this equation holds then the Mpemba effect cannot be real. Say the two cups start at 30℃ and 5℃. When the warmer one cools down to 5℃, it will have to follow the same route to freezing that its rival has already begun, but without the headstart. This is the nub of the problem: the warm cup doesn't just have to cool faster initially, it has to *overtake* its rival. When it reaches 5℃, it has to be cooling faster than the cooler cup was at that same temperature, so the equation above cannot be true. The rate of change of temperature can't just depend on the current temperature T; it must also depend on T_0, the initial temperature. So in fact our model should be

$$\frac{\mathrm{d}T}{\mathrm{d}t} = f(T, T_0).$$

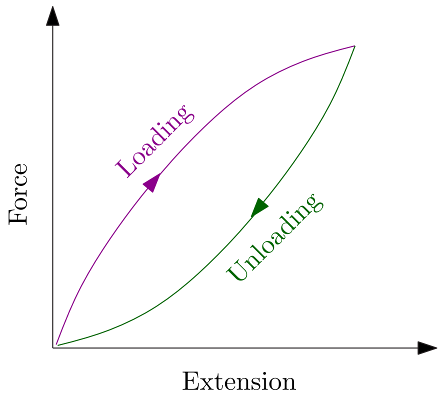
This is therefore a system with memory. What's happening in the cup doesn't just depend on its present temperature, but also on its past temperature, on its history. This is the key point. Any successful theory must explain not only how the warmer cup cools faster initially, but moreover how it can overtake. It must explain why the system has memory.



One possibility for the temperatures of the water in the two cups over time

Hysteresis and systems with memory

The idea of a physical system having memory can be quite counterintuitive, but many examples do exist. Imagine hanging weights on an elastic band. When you add a new weight it might stretch from 10 cm to 12 cm, say. But if you remove that new weight the band doesn't return all the way to 10 cm, it stays a bit more stretched. This is an example of *hysteresis*—when changing an input then reversing the change doesn't bring the output back to the original level. Hysteresis is an interesting feature found in systems with memory.



Force vs extension for an elastic band. This loop-shaped graph is the signature of hysteresis.

Many other important examples of hysteresis can be found. Applying a magnetic field to iron magnetises it, but removing the field doesn't reverse the process. This is known as magnetic hysteresis and is essential for hard disk drives to work. In flight, when an aeroplane flies up at too steep an angle, it starts to stall and lose lift. It can recover by decreasing its angle of attack, but it has to go down to a lower angle than that at which stalling started. And it's not only physical systems that can exhibit hysteresis. During a recession, unemployment typically increases. But when the economy recovers, the employment rate doesn't necessarily recover with it.

A worrying possibility is that our climate may exhibit hysteresis. Beyond certain tipping points, changes may be irreversible. This is obviously true for the extinction of animal species but might also apply to, for example, the melting of major ice sheets.

Theories

So if the Mpemba paradox requires the system to have memory, which current theories incorporate this feature?

One idea is that the warm water evaporates more, so with a lower volume it can cool quicker. If V is volume then we can write

$$\frac{\mathrm{d}T}{\mathrm{d}t} = f(T, V),$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = g(T, V),$$

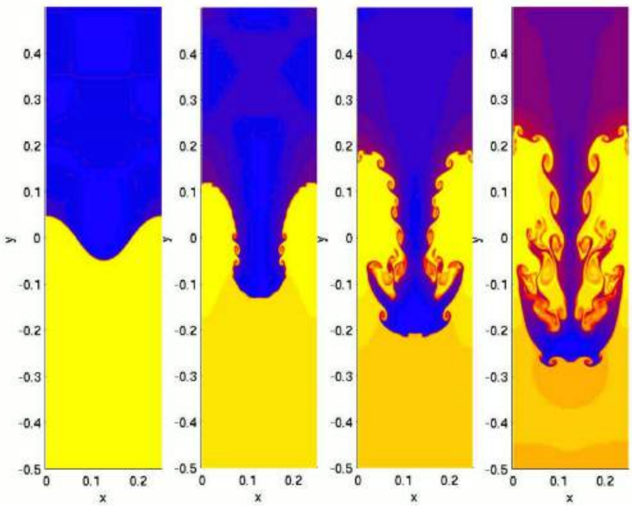
for functions f and g. Thus memory is stored in the extra variable V. There is good evidence that this effect has a role but there are also control experiments with sealed lids or no loss of mass that show that it cannot be the sole explanation.

Alternatively, there are a whole group of explanations asserting that the environment makes the difference. One possibility is that the warmer cup might melt through frost at the bottom of the freezer until it touches the actual freezer surface, a better conductor of heat. Here, an additional variable E representing the environment stores memory. These explanations are certainly plausible but are also easy to control for in experiments. It seems unlikely that one of them could be the cause of the Mpemba paradox in every single case.

Perhaps the most successful explanation is related to supercooling. Supercooling is when a liquid passes below its freezing point, but doesn't yet freeze because there are no places, such as dust particles, for the ice to starting forming. The liquid wants to freeze, but doesn't know how to. Some experiments suggest that the cool water supercools more than the warm water. This would explain the Mpemba paradox, but replace it with a new puzzle: why does cooler water supercool more? This idea also highlights the need for a precise definition of the Mpemba paradox—does the warm water have to freeze first or reach 0℃ first? The supercooling theory can only explain the former.

Convection

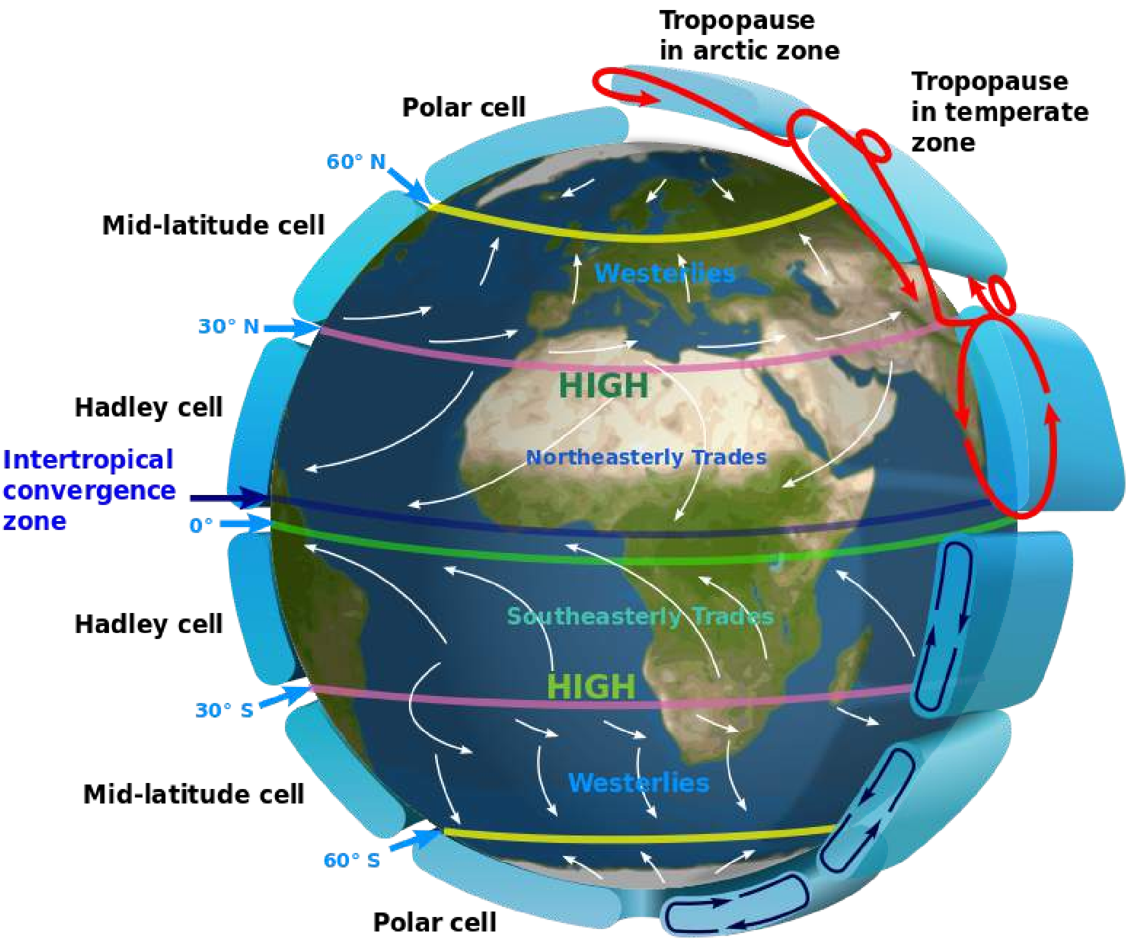
A final major theory that could explain the memory in the Mpemba experiment is based on convection. Convection is when a liquid or gas mixes because of density differences: hot air is light and rises to the top, cold air is heavy and sinks to the bottom. Convection is the key driver of our weather. The atmosphere is heated at the equator and cooled at the poles, which drives a huge conveyor belt of air.



Los Alamos National Laboratory[1]

A mathematical simulation of a 'Rayleigh-Taylor instability', an experiment with cold liquid (blue) starting above warm liquid (yellow). Note the beautiful spiral patterns, known as 'Kelvin-Helmholtz instabilities', that form at the interface.

A particularly beautiful example of convection can be seen if you take a pan of liquid such as oil and heat it from below at the right rate. Convection cells in the shape of hexagonal prisms known as ‘Rayleigh-Bènard cells’ form. Hot oil rises in the centre of each hexagon, and cooler oil sinks at its edges. It is remarkable that the molecules of oil arrange themselves into this ordered and efficient mathematical pattern. Although not the standard explanation, it has been speculated that this may be the origin of the Giant’s Causeway, a set of 40,000 polygonal basalt columns in County Antrim, Northern Ireland. These incredible shapes were formed as the molten basalt cooled.



A cartoon of the convection cells that drive the atmosphere

A convecting liquid can lose heat faster, which may explain the Mpemba paradox. Convection moves hot water to the top so, if this is where the heat is lost, the cup would cool faster. This shows that the warmer cup could cool faster initially, but it may also explain how it can overtake. Convection is known to exhibit hysteresis. Water at 5℃ in a -20℃ freezer might not start convecting, but if it were already convecting, it could continue. So the memory is stored in the level of convection in the cup.

Understanding convection requires more complex equations than we have used so far. We would need to look not just at the average temperature, but at the temperature and velocity at every point in the cup. Interpreting and solving these equations requires more sophisticated mathematics, but is certainly possible.

Uni. Iowa, Physics and Astronomy

Left: An experiment with oil heated from below. Hexagonal Rayleigh-Bénard cells have formed.
Right: The Giant's Causeway in Northern Ireland.

A solution to the paradox?

It seems strange that the Mpemba paradox hasn't yet been resolved. Surely a set of comprehensive experiments could test each one of the major theories? For example, you could test the role of convection by stirring both cups or the role of melting frost by placing them on an insulating mat. Modern experimental techniques can measure the heat and even the velocity at many points within the liquid. So we should be able to build a detailed picture of what is happening.

Perhaps the reason the paradox hasn't been resolved is a practical one: it crosses so many disciplinary boundaries. A mathematician views it mathematically, whilst a chemist looks for chemical solutions. A full resolution requires the input of many disciplines.

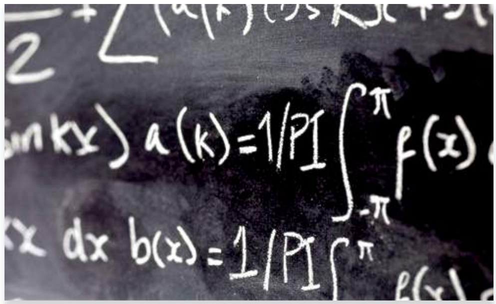
Or perhaps there is no single solution. Maybe both supercooling and convection are strong enough effects to individually cause the Mpemba paradox. Then results from different experiments would seem incompatible and confusing to a scientist assuming a single cause. Either way, the Mpemba paradox remains a fascinating scientific mystery. It is simple, seemingly impossible and links together exciting science across the disciplines, from supercooling to convection to the bizarre property of hysteresis.

*Oliver Southwick is a PhD student from the UCL Department of Mathematics modelling large scale ocean currents. He can be contacted on Twitter at **@oliversouthwick**.*

√Roots

In our new history column, Emma Bell explores some of the lesser known stories from the bygone days of mathematics...

All things being equal



Imagine the scene: the year is 1557. Henry VIII's eldest daughter, Mary, is on the English throne. It'll be another year before her younger sister, Elizabeth, becomes queen. You've published a fair few mathematical texts, and you're halfway through writing your latest book *The Whetstone of Witte*, the second in a pair of books on Arithmetic.

You're determined to only use English language in the book but you are getting really frustrated with having to write "is equal to" every time you note down an equation. Then it dawns on you! Why not use a symbol to represent "is equal to"? It'll save time. It'll save ink. After all, isn't mathematics all about efficiency? But what symbol to use?

It occurs to you: no two things are more equal than two parallel lines, so to "avoid the tedious repetition" of the words "is equal to", you introduce the symbol "=". You have invented the equals sign.

We've all been there—LOL, BRB, </3—finding short ways to express yourself to save time and effort.

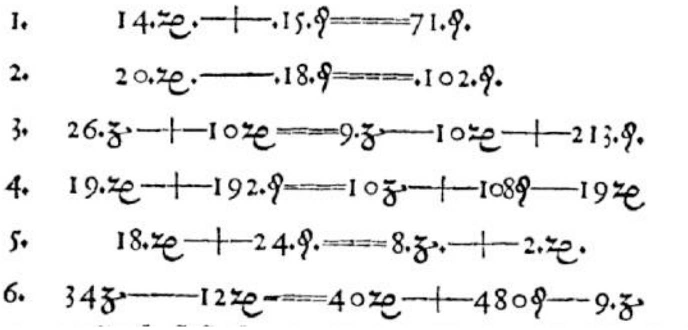
Robert Recorde

The equals sign was invented as shorthand by Robert Recorde, a Tudor mathematician and doctor who published widely on a variety of topics. He was born in Tenby, Wales in around 1510, and died in Southwark in 1558 whilst in prison for not paying a fine.

During his life, he took on many roles. He was in charge of the Royal Mint and taught mathematics at both Oxford and Cambridge universities. He was a qualified medical doctor, and worked as the court physician for both Edward VI and Mary. The following is an excerpt from the original text of *The Whetstone of Witte*, published in 1557:

> *"Howbeit, for easy alteration of equations. I will propound a few examples, because the extraction of their roots, may the more aptly be wrought. And to avoid the tedious repetition of these words: is equal to: I will set, as I do often in work use, a pair of parallels, or Gemowe [twin] lines of one length, thus: =, because no two things, can be more equal."*

Recorde initially used this new symbol to set up a system of six equations:



The symbols within the equations have meanings, and Recorde provides us with a key:

 - an absolute number,

 - an unknown, or root,

 - a square number.

We can rewrite Recorde's equations to make them look more familiar. Here are the first three:

$$14x + 15 = 71$$
$$20x - 18 = 102$$
$$26^2 + 10x = 9^2 - 10x + 213.$$

It is easy to appreciate how useful this new symbol was. Up until Recorde's notation, the most common way to write "equal to" was to use the Latin "aequales" or "aeq". Recorde was very keen on promoting English vocabulary in an English mathematics textbook. He believed that everyone should be able to access mathematics. The + and − signs had not been used in England previously, and with Recorde's invention of =, the equations included in his book look like they could have been written this week, not 450 years ago. Robert Recorde was a trailblazer.

However, whilst abbreviations in common use currently spread like wildfire, the spread of Recorde's equals sign was much more sedate. In 1631, three separate books were published—including *Trigonometria*, the influential book by Richard Norwood—all using the parallel lines equals sign. Sir Isaac Newton (1642–1727) then picked up on the notation and used it in his works, and so on, until it became the ubiquitous sign that mathematicians now use on a daily basis. This was hardly a viral spread with the scale or speed of "FTW" or

"This was hardly a viral spread with the scale or speed of "FTW" or the hashtag, but the slow and steady acceptance of = as the universal sign of equality is clear to see."

the hashtag, but the slow and steady acceptance of = as the universal sign of equality is clear to see.

Indeed, the Human Rights Campaign unveiled the equals sign as their logo in 1995, and it has become synonymous with the LGBT community's equal rights movement. Inscribed on a memorial plaque to Recorde in St Mary's Church, Tenby, is the line "invented the sign of equality", and I am astounded how prophetic this statement turned out to be.

Red Human Rights Campaign logo

In *Mathematics* by David Eugene Smith (1923), Recorde is referred to as the founder of the British School of Mathematics, but I hadn't heard of him until I embarked on the research for this article. That needs rectifying. We now know that when it comes to impact and influence on the brevity and efficiency of mathematics, Robert Recorde is without equal.

Emma Bell is a maths teacher at Franklin College in Grimsby, UK. You can follow her on Twitter ***@El_Timbre****, and email her at* ***emma.bell@franklin.ac.uk****.*

Robert Recorde's book, The Whetstone of Witte, *1557 (the title was a pun about sharpening your mathematic wits) is available to download for free on* ***archive.org****.*

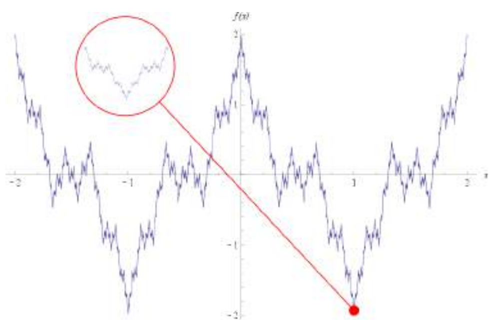
My Favourite Function

Weierstrass Function

Anna Lambert

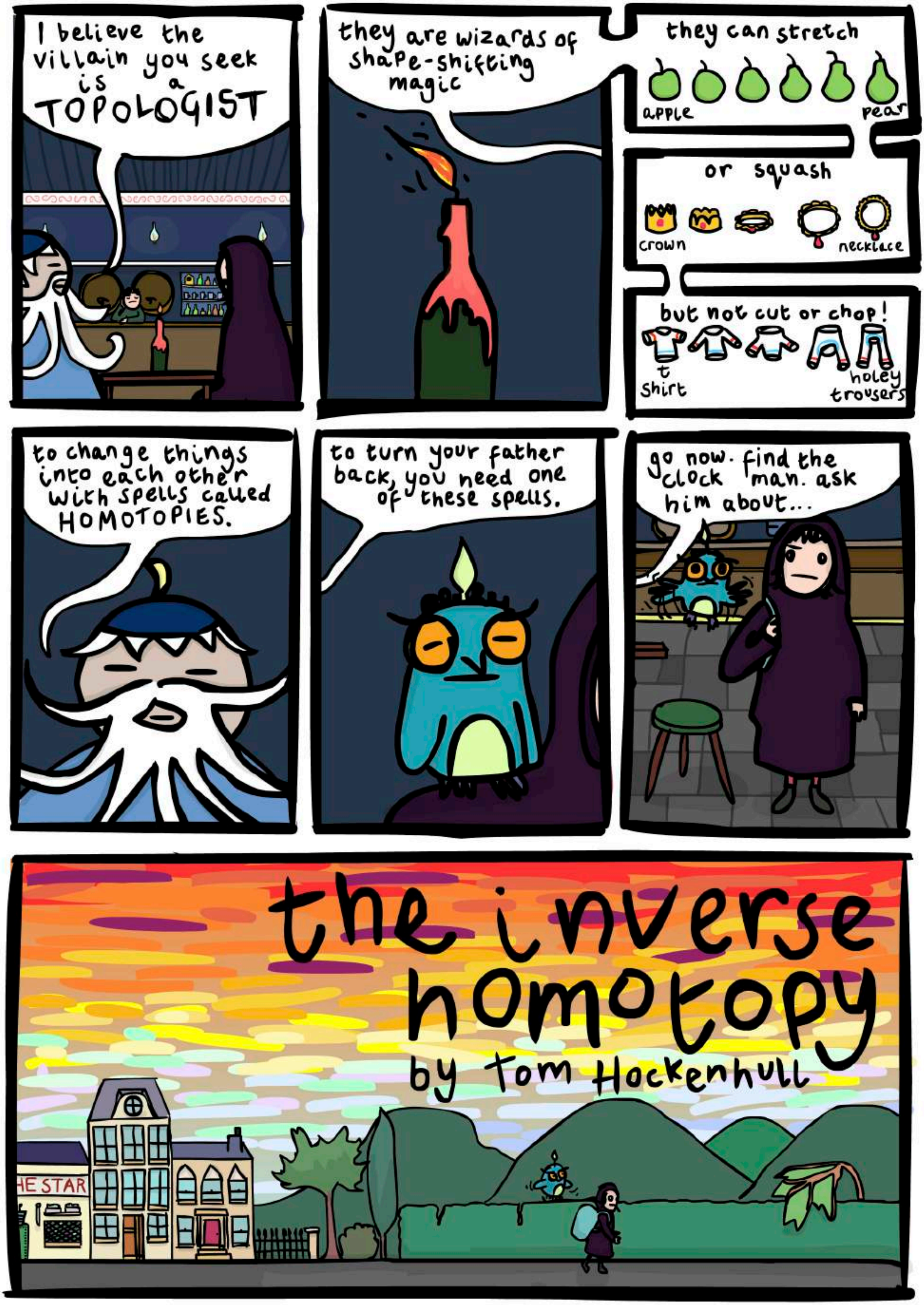
My favourite function has infinitely many zigzags. It's called the Weierstrass function, and is a classic feature of a first term analysis course. It's written

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$



where $0 < a < 1$, b is a positive odd integer and $ab < 1 + \frac{3}{2}\pi$. It might not look like much, but it was the first known example of a function that is continuous but nowhere differentiable. What does that mean? Well, a function is continuous if you can draw it without taking your pen off the paper. It is differentiable if the slope of the function varies smoothly, but it will fail to be differentiable at a point if that point is sharp. For example, a zigzag is continuous, and differentiable everywhere except for at its zigs and zags. So for a function to be nowhere differentiable, every single point on the curve must be sharp. This is an incredibly weird concept to think about. Clearly all of these sharp points cannot be visible at once, but as you zoom in, you can see more and more zigs and zags. This is just like a fractal—as you magnify, the curve looks the same and reveals even more detail.


tell me your story, child
Well... I came home to find...
my dad
had been turned into an owl
So now I want to find-
WHOOO
yes, whoo did this, and how to turn it back!
hmmmmmmmm
hmmmmmmmmm...



I believe the villain you seek is a TOPOLOGIST
they are wizards of shape-shifting magic
they can stretch
apple
pear
or squash
crown
necklace
but not cut or chop!
t shirt
holey trousers
to change things into each other with spells called HOMOTOPIES.
to turn your father back, you need one of these spells.
go now. find the clock man. ask him about...
the inverse homotopy
by Tom Hockenhull
HE STAR

The perils of *p*-values

Why more discoveries are false than you thought

David Colquhoun

Imagine that we have a group of 20 volunteers. We give all 20 people identical pills, and measure a response in each of the people. The responses would not all be the same—there is always some variability. If we divide the 20 responses randomly into two groups of 10, the means of the two groups will therefore not be identical.

If we had instead given each group of 10 people different pills (say drug A and drug B) then we would also find that the means of the two groups differed. If drug A was better than B then the mean response of the 10 people given A would be bigger than the mean of the 10 responses to B. But of course the response of group A might well have been bigger, even if drugs A and B were actually identical pills.

It is one of the jobs of applied statisticians to tell us how to distinguish between random variability and real effects. They can tell us how big the difference between the means for A and B must be before we believe that A is really better than B and not just the result of random variability.

It is the aim of this article to persuade you that the ways of doing this that are commonly taught give rise to far more wrong decisions than most people realise. This is not trivial. It gives rise to the publication of discoveries that are untrue. For example, it may result in the approval of medical treatments that don't work.

How to tell whether an effect is real, or mere chance

In the example above, the 10 people who were given drug A were chosen at random from the 20 volunteers. If the two drugs were in fact identical then each of the 20 people would have given the same response regardless of whether they had been allocated to the A group or the B group. The response would be a characteristic of the person, and not dependent on whether they got pill A or pill B. So the observed difference between means would depend only on which particular individuals were allocated to group A or B, i.e. on how the random numbers came up. Therefore it makes sense to look at the outcomes that would have been observed if the random numbers had been different.

There are 184,756 ways of selecting 10 observations from 20, giving 184,756 differences between means that are what we would expect to observe if in fact the treatments were identical. The speed of computers is such that they can all be inspected in just a few seconds. Figure 1 shows the distribution of all 184,756 differences between means. Since it is based on the premise that the treatments were identical, the average difference between means is zero.

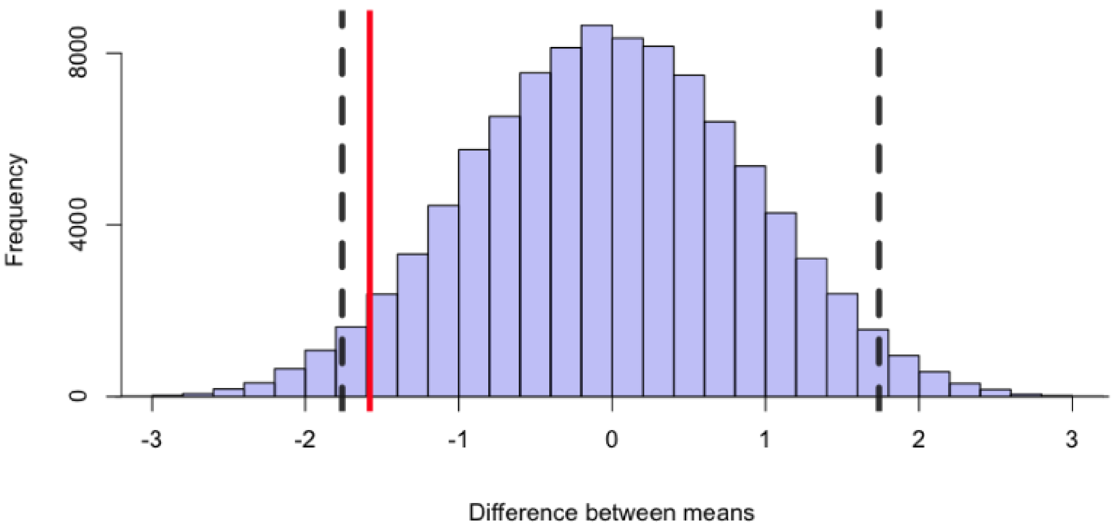


Figure 1: A randomisation distribution. It shows the distribution of all 184,756 differences between means for every possible way of choosing 10 observations from 20, based on the assumption that the pills are identical, so the mean of the distribution is zero. The vertical dashed lines mark 2.5% of the area in the lower tail and 2.5% in the upper tail. The red line marks the difference between means that was observed in the experiment. Because the red line lies between the dashed lines we can conclude that there is not strong evidence to say that the true difference is not zero.

The observed mean response of the 10 people on drug A was 0.75, and the mean for drug B was 2.33. So the observed difference between means was $0.75 - 2.33 = -1.58$. This value is marked by the vertical red line in Figure 1. About 4% of the differences are below the observed value, -1.58. Another 4% are above $+1.58$. So we find that if the two drugs were identical there would be a probability of $p = 0.08$ of finding a difference between means, in either direction, as big as that observed, or even bigger. This 8% is termed the p-value. If it is small enough we reject the premise that the two treatments are identical. And 8% is not really very small: the experiment doesn't provide strong evidence against the idea that the drugs are identical. This procedure is known as a randomisation test.

William Sealy Gossett ('Student')

This sort of problem would be more commonly analysed using a Student's t-test. This test was invented at UCL in 1908, by William Sealy Gossett, who wrote under the pseudonym 'Student'. He was chief brewer at Guinness. On a visit to UCL, to work with Karl Pearson, he derived the first test of significance that was valid for small samples and the data used in Figure 1 is from a paper by Cushny & Peebles (1905) and was later used in the paper that first described the t-test (Student, 1908). (Cushny was the first professor of pharmacology at UCL.) It was pioneering work, but it should now be replaced by the randomisation test described here, which makes fewer assumptions. In this case, the samples are sufficiently large that the result of the t-test ($p = 0.079$) is essentially the same as what we obtained.

The postulate that the treatments are identical is called the null hypothesis, and this approach to inference—attempting to falsify the null hypothesis—has been the standard for over a century. It's perfectly logical.

What could possibly go wrong?

The problems of null hypothesis testing

The p-value does exactly what it claims. If it is very small, then it's unlikely that the null hypothesis is true. Falsifying hypotheses is how science works. Every scientist should be doing their best to falsify their pet hypothesis (the fact that many don't is one of the problems of science, but that's not what we are talking about here).

So how small must the p-value be before you can reject the null hypothesis with confidence? A convention has grown that $p = 0.05$ is some sort of magic cut off value. If an experiment gives $p < 0.05$ the result is declared to be "significant" with the implication that the effect is real. If $p > 0.05$ the result is labelled "not significant". This practice is almost universal among biologists, despite being obvious nonsense—clearly the interpretation of $p = 0.04$ should be much the same as $p = 0.06$.

"The real problem lies in the fact that the p-value doesn't answer the question that most experimenters want to ask."

But that is only the beginning of the problems. The real problem lies in the fact that the p-value doesn't answer the question that most experimenters want to ask. What I want to know is "if I claim, on the basis of my experimental results to have made a discovery, how likely is it that I'm wrong?". If you claim to have made a discovery (like drug B works better than drug A), but all you are seeing is random variability, then you make a fool of yourself. And the aim of statistics is to prevent you from making a fool of yourself too often.

If you ask most people what the p-value means, you'll very often get an answer like "it's the probability that you are wrong".

It isn't, and I'll explain why.

The crucial point is that the p-value only tells you about what you'd expect if the null hypothesis were true. It says nothing about what would happen if it wasn't true. Paraphrasing the words of Sellke et al. (2001),

> *"Knowing that the data are 'rare' when there is no true difference is of little use unless one determines whether or not they are also 'rare' when there is a true difference."*

Let's define the probability that we are wrong if we claim an effect is real as the false discovery rate (or the false positive rate). That is what we want to know, but it is quite different from the p-value, and it is less straightforward to calculate. In fact it is impossible to give an exact value for the false discovery rate for any particular experiment. But we can make the following statement:

If we declare that we have made a discovery when we observe $p = 0.047$, then we have at least a (roughly) 30% chance of being wrong.

In other words, when we use $p = 0.05$ as a criterion for declaring that we have discovered something, we'll be wrong far more often than 5% of the time. That alone must make a large contribution to the much-publicised lack of reproducibility in some branches of science. The paper by Stanford epidemiologist, John Ioannidis, *Why most published research findings are false* touched a nerve. It has been cited over 3,000 times. Of course, he wasn't talking about all science; just about some parts of biomedical research.

Why p-values exaggerate the strength of evidence

Take the simplest possible example. Let's ask what the false discovery rate is if we do a single test and obtain the result $p = 0.047$. Many people would declare the result to be (statistically) significant, and claim that the effect they were seeing was unlikely to be a result of chance.

We can treat the problem of significance testing as being analogous to screening tests, which are intended to detect whether or not you have some illness. In screening we need to know about false positive tests—the fraction of all tests that say you are ill when you are not—because it is distressing and expensive to be told you are ill when you're not.

> *If only 1% of the population suffer from the disease then a staggering 86% of positive tests are wrong!*

There are three things that need to be specified in order to work out the false positive rate for a screening test: these are the specificity of the test, the sensitivity of the test, and the prevalence of the condition you are trying to detect in the whole population that you're testing. The specificity of a test is the percentage of negative results identified correctly. For example, a screening test with a specificity of 95% means that 95% of people who haven't got the disease test, correctly, negative. The remaining 5% are false positives: the test says that they have the disease whereas in reality they don't. That sounds quite good. The sensitivity of the test is the percentage of positive results identified correctly: you have a disease and the test agrees. So, if the sensitivity is 80%, then if you have the disease, you have an 80% chance that it will be detected correctly. That also doesn't sound too bad.

However, if the condition that you are trying to detect is rare (the prevalence of that condition), then most positive tests will be false positives. For example, the screening test for mild cognitive impairment (which may or may not lead to Alzheimer's disease) has the specificity and sensitivity given above, but if only 1% of the population suffer from the disease then a staggering 86% of positive tests are wrong! This happens because most people haven't got the disease and so the 5% of false positives from them overwhelms the small number of true positives from the small number of people who actually have the disease. This is why screening for rare conditions rarely works.

Now we can get to the point. How does all this apply to tests of significance? Figure 2 shows an argument that's directly analogous to that used for screening tests.

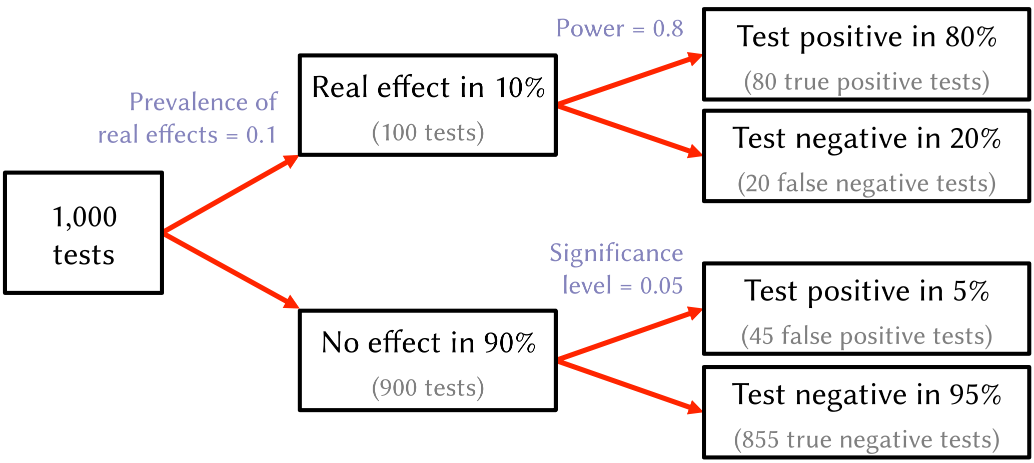


Figure 2: Tree diagram to illustrate the false discovery rate in significance tests. This example considers 1,000 tests, in which the prevalence of real effects is 10%. The lower limb shows that with the conventional significance level, p=0.05, there will be 45 false positives. The upper limb shows that there will be 80 true positive tests. The false discovery rate is therefore 45/(45+80)=36%, far bigger than 5% (from Colquhoun, 2014).

Again we need three things to get the answer. The first two are easy. The probability of getting a false positive when there is no effect is simply the significance level, which we have seen is normally set to 0.05. It's the same thing as (1 − specificity) in the screening test example. The power of the test is the probability that we'll detect an effect when it's really there. It's the same thing as the sensitivity of a screening test. It depends on the variability and the size of the effect. The sample size is customarily chosen to give a power of 0.8 (as in Figure 2) though it's very common for sample sizes to be too small, so the power of many published tests is actually in the range 0.2–0.5.

The third thing that we need is the tricky one. In order to work out the false discovery rate, we need an analogue of the prevalence in the screening test example. In the case of screening, that was simply the proportion of the population that suffered from the condition. In the case of significance tests, the prevalence is the proportion of tests in which there is a real effect, i.e. the null hypothesis is false. If one were testing a series of drugs, you'd be very lucky if the proportion that worked was as high as 10%; so, for the sake of an example, let's take the prevalence to be 0.1.

We can now work though the example in Figure 2. If you do 1,000 tests, then our prevalence means that in 900 (90%) of them the null hypothesis is true (there is no effect) and in 100 of them (10%) there is a real effect. Of the 900 tests in which there is no real effect, applying the significance level of 5% means that 45 of them will give $p < 0.05$: they are false positives and we'd have claimed to have found a result that isn't there. That is as far as you can get with classical null hypothesis testing. But to work out what fraction of positive tests are false positives we need to think about not only what happens when the null hypothesis is true (the lower arm in Figure 2), but also what happens when it is false (the upper arm in Figure 2). The upper arm has 100 cases where there is a real effect and, due to the power (or sensitivity of the test), 80% of these are detected (i.e. give $p < 0.05$), so there are 80 true positive tests.

Therefore the total number of positive tests is 45 + 80 = 125, of which 45 are false positives. So the probability that a positive test is actually false is 45/125 = 36%. This is far bigger than the 5% significance level might suggest.

> *If you claim a discovery on the basis that p = 0.047 you'll be wrong at least 26% of the time and maybe much more often.*

The argument in Figure 2 shows that there is a problem, but it still doesn't quite answer the original question: what's the false discovery rate if we do a single test and obtain the result $p = 0.047$? To answer that, we need to look only at those tests that give $p = 0.047$ (rather than all tests that give $p < 0.05$, as in Figure 2). This is easy to do by simulation (see the R script on our website). The result is that if you claim a discovery on the basis that $p = 0.047$ you'll be wrong at least 26% of the time (that's for a prevalence of 0.5), and maybe much more often. For a prevalence of 0.1 (as in Figure 2) a staggering 76% of such tests will be false positives.

What is the prevalence of true effects?

The biggest problem with trying to estimate the false discovery rate is that the prevalence of true effects is unknown. It is what a Bayesian would call the prior probability that there is a real effect ('prior' meaning the probability before the experiment has been done). As soon as the word Bayes is mentioned, statisticians tend to relapse into arguing amongst themselves about the principles of inference: this is unhelpful to experimenters. In my opinion, it is not acceptable (in the absence of strong empirical evidence) to assume that your hypothesis has a chance of being true that's more than 50% before the experiment has been done. If you did then it would be tantamount to claiming that you had made a discovery and justifying that claim by using a statistical argument that assumed that you were likely to be right before you even did the experiment! Most editors and readers would reject such an argument, but they are happy to accept marginal p-values as evidence.

This alone may explain why so much research has proved to be wrong.

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*David Colquhoun FRS is a professor of pharmacology at UCL, and a prolific critic of pseudoscience and scientific fraud. His website, DC's improbable science (**dcscience.net**) won first prize in the 2012 Good Thinking Society awards for science blogs. You can follow him on Twitter **@david_colquhoun**.*

*For more information, please refer to the original paper at **rsos.royalsocietypublishing.org/content/1/3/140216**, visit David's website or go to **chalkdustmagazine.com** where you can find both a video in which he explains in greater detail the dangers of p-values and two pieces of R code written by David which you can use to run your own simulations to convince yourself of the risks of using the p-value when reporting results.*

References and further reading

Colquhoun D (1971). *Lectures on Biostatistics* Clarendon Press, Oxford (downloadable at dcscience.net).

Colquhoun D (2014). An investigation of the false discovery rate and the misinterpretation of p-values. *R Soc Open Sci* **1**, 140216.

Cushny AR & Peebles AR (1905). The action of optical isomers: II. Hyoscines. *J Physiol* **32**, 501–510.

Ioannidis J (2005). Why most published research findings are false. *PLoS Med* **2**, 696–701

Sellke et al. (2001). Calibration of p-values for testing precise null hypotheses. *The American Statistician* **55**, 52–71.

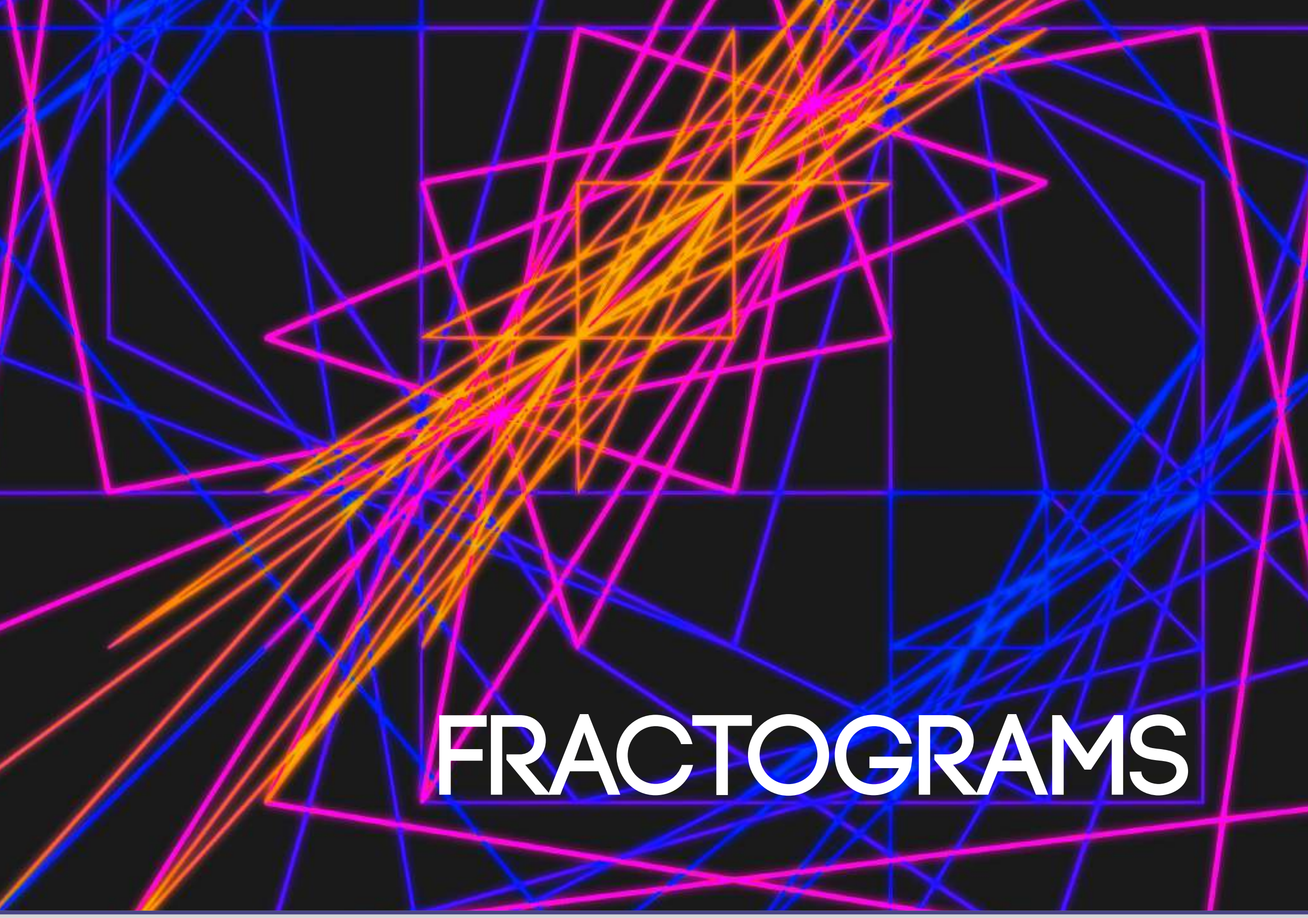
Senn S & Richardson W (1994). The first t-test. *Stat Med* **13**, 785–803.

Student (1908). On the probable error of a mean. *Biometrika* **6**, 1–25.

Odd Squares

How many square numbers are there whose digits are all odd?

Answers at chalkdustmagazine.com/answers



FRACTOGRAMS

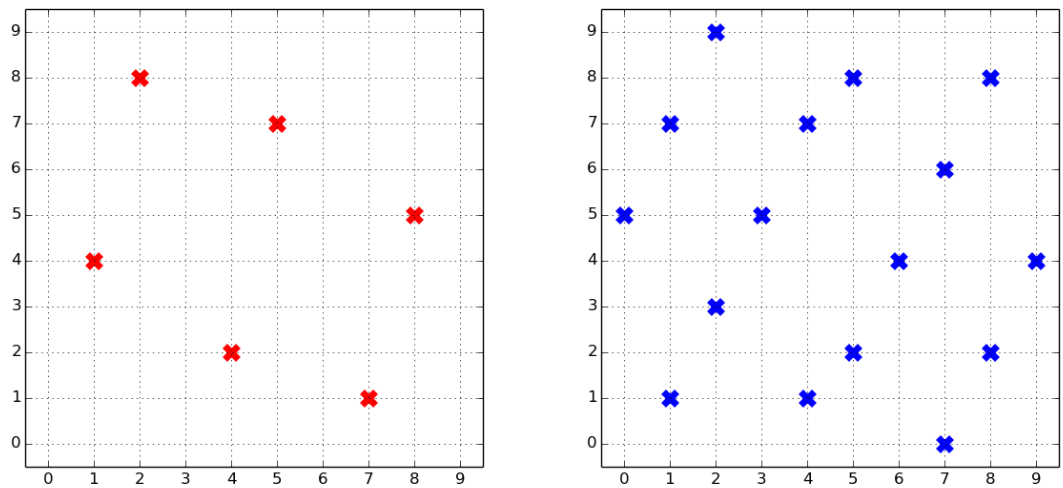
Hugh Duncan

Bored one day in a staff meeting, I took to playing around with numbers—a nice way to pass the time. I wondered if chaotic numbers might exist; that is, numbers whose digits at first might look quite random, but hidden within this apparent disorder would be the signature of order that lies at the heart of chaos. I had big notions that maybe the digits making up famous irrational numbers like π or $\sqrt{2}$ might be such numbers, but I decided to start with more simple numbers, those of the recurring decimals. I took the fraction $1/7$ as my starting point.

The first return maps

Taking the decimal expansion of the fraction $1/7 = 0.142857\ldots$, I used consecutive pairs of digits as coordinates and plotted them on a simple *x*-*y* grid. This gives $(1,4)$, $(4,2)$, $(2,8)$, $(8,5)$, $(5,7)$, then back to $(7,1)$ before the pattern repeats. Note that I use each digit twice. This is the essence of a return map: one reading relates to the previous and, in the case of chaos, data that might initially appear to be very random will show unexpected order on the graph. A return map of the digits of a decimal fraction I shall call a *fractogram*. See overleaf for the unconnected fractogram for the fraction $1/7$—unconnected as the points are not joined together.

Note that the six points make two approximately straight, parallel lines. The shape of the six points also has rotational symmetry of order two. Next to it is the unconnected fractogram for $1/17$. It also has sets of points making roughly straight parallel lines and has rotational symmetry of order two. Not all fractions give this result of course. For example, $1/3$ being $0.3333\ldots$ gives



Unconnected fractograms for $1/7 = 0.142857...$ (left) and $1/17 = 0.0588235294117647...$ (right)

only one point $(3, 3)$; while $1/11$ being $0.0909\ldots$ gives two points $(0, 9)$, $(9, 0)$; and $1/37$, which is $0.027027\ldots$, gives three points and so on. Some decimal fractions truncate, such as $1/8$, which is 0.125, so after plotting the points $(1, 2)$ and $(2, 5)$ it comes to an end. It is left as an exercise for the reader to try others and discover the wide variety of patterns!

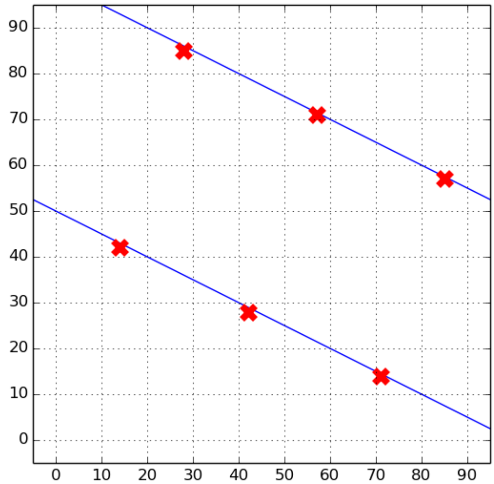
Raising the resolution

The almost straightness of the lines formed by the coordinates on the $1/7$ and $1/17$ fractograms intrigued me. Why were they almost straight? Why were they not completely straight? I decided to play another game. What would happen if instead of plotting consecutive pairs of digits as coordinates, I plotted consecutive pairs of double digits? Taking the fraction $1/7$ again, I obtained $(14, 42)$, $(42, 28)$, $(28, 85)$, $(85, 57)$, $(57, 71)$ and $(71, 14)$; which, when plotted on a fractogram, gave the pattern shown on the next page.

Taking consecutive pairs of the decimal expansion of $1/7$ and plotting them as pairs of double-digit coordinates has made the six points lie closer to two straight lines. Well, what if triplets of digits were used, giving the coordinates $(142, 428)$, $(428, 285)$, $(285, 857)$, $(857, 751)$, $(751, 514)$ and $(514, 142)$? The six points lie even closer to the straight lines they appeared to follow in the previous approximate cases. Indeed, it can be shown that if this process is continued the points would eventually lie exactly on one of two lines, each of gradient $-1/2$. Whether that was to be expected or not, it is curious to see that the gradient of the lines is a very simple fraction. Why should that be? For the $1/17$ fractogram, the gradient of the lines approaches $3/2$.

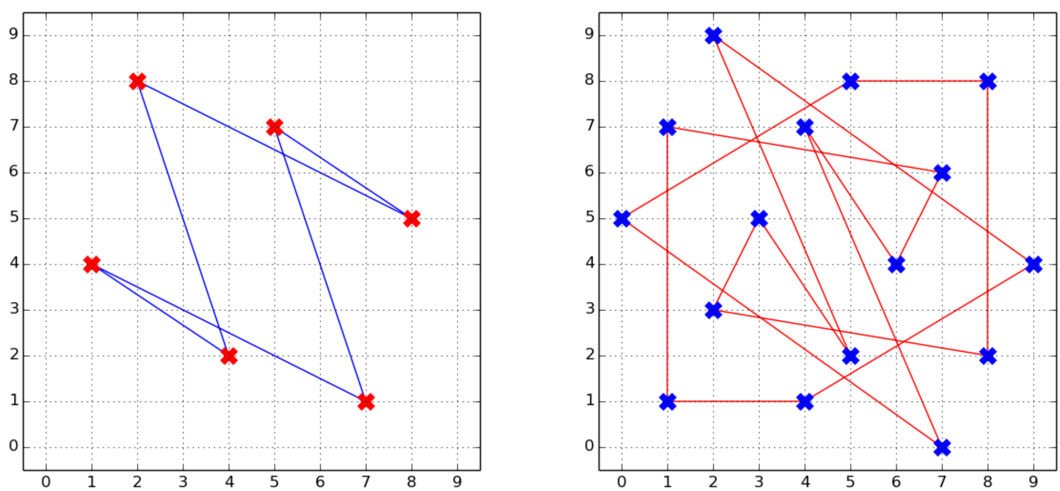
Joining the dots

The next step in the experiment was to see if there were any further patterns hidden in these almost orderly sets of points. I wondered if the order in which they appeared on the graph was



Unconnected fractogram for 1/7 with double digit coordinates

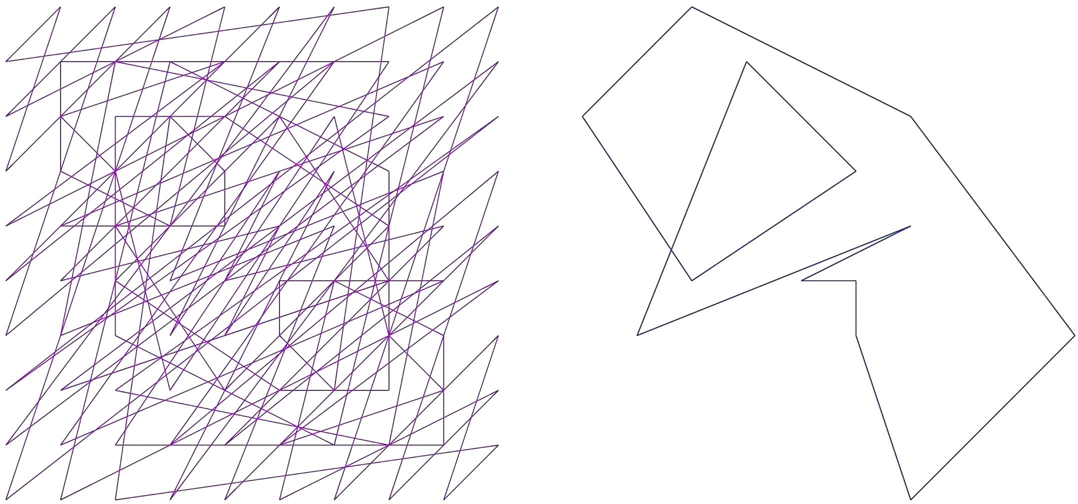
relevant, so I did the equivalent of the children's picture-drawing pass time of 'joining the dots' for the fraction 1/7. I call this a *connected fractogram*. Doing this for our original fractogram, where the coordinates were given by a single digit, results in a curious shape. It has the same rotational symmetry of order two but the lines do not follow the original approximately straight ones:



Connected fractograms for $1/7 = 0.142857\ldots$ (left) and $1/17 = 0.0588235294117647\ldots$ (right)

The connected fractogram of $1/17$ is also shown above and it too has a symmetrical set of lines that have nothing to do with the rough straight lines that appeared at the beginning.

Each fraction on a connected fractogram has its own pattern. Some are symmetrical shapes like $1/7$, the lines of which do not usually follow those approximately straight lines from the first experiment at all. Some are simple polygons: $1/37$ is a triangle, while $1/101$ is a square. Some merely show a random set of connected lines making a closed shape. To assist this process, which done by hand is slow, my son, Fabien Duncan, kindly wrote a program that does the job in a fraction of the time. See below for just two examples (1/97 and 2/79) with the grid lines removed. Doesn't one of them look a little like Bart Simpson?!



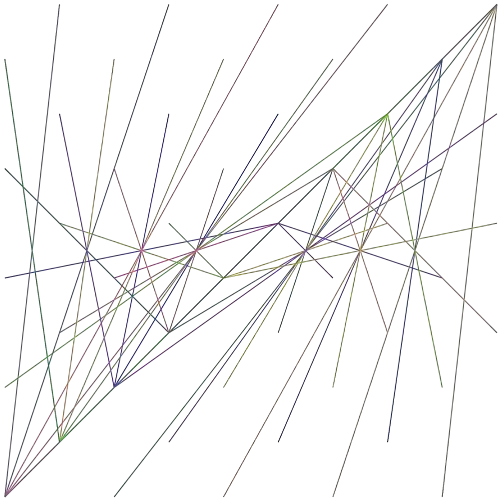
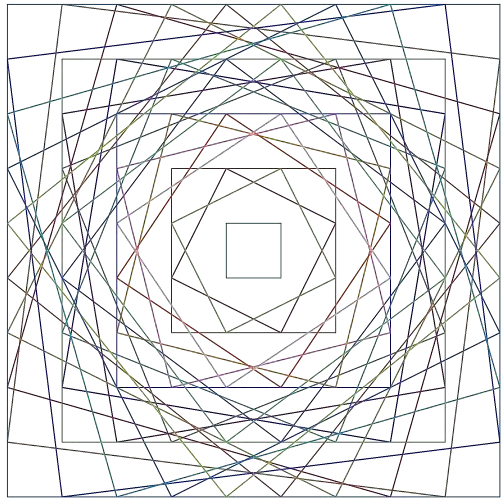
The connected fractograms 1/97 (left) and 2/79 (right)

Superposed fractograms

If one takes the fraction 1/55, which is 0.018181 . . . , it only has 3 different digits, hence only three points on a fractogram. By increasing the numerator by one to 2/55, one gets 0.03636 . . . : a different set of three points. By themselves these fractograms are not so interesting but if one were to take all of them—1/55, 2/55, 3/55, up to 54/55—and plot them all on the same graph then one obtains a more fascinating pattern, which I call the *superposed fractogram* $n/55$ (see next page). $n/55$ has loose ends and is not cyclic but it almost has rotational symmetry. Next to $n/55$ is the superposed fractogram for $n/101$: it is made up of squares and has rotational symmetry of order four plus line symmetry of equal complexity!

Where do we go from here?

This is only the beginning and there are so many other areas to explore and beautiful patterns still to discover! For example, we could try to tessellate the unconnected fractograms, placing them next to one another like bathroom tiles to see what large scale pattern they make. Or we could investigate what other convex polygons one could make with connected fractograms. How many of these are regular? What shapes are possible? Pentangles? Steps? Spirals? Are any shapes impossible? We could make an excursion into other bases aside from the base 10 that we are so used to. What would our fractograms look like in a different base, like base 7 or base 13? Or let's extend our fractograms into three dimensions, representing our friend 1/7 by the six points $(1, 4, 2)$, $(4, 2, 8)$, $(2, 8, 5)$, $(8, 5, 7)$, $(5, 7, 1)$ and $(7, 1, 4)$. Perhaps we'd like to make a movie by creating a

The superposed fractograms $n/55$ (left) and $n/101$ (right)

series of fractograms for a given denominator d $(1/d, 2/d, \ldots, (d-1)/d)$ and then playing them one after another to see how the shape evolves. Or... the list goes on; we are limited only by our own imagination.

And in case you're wondering: no, there is no hidden pattern to the digits of π when plotted as a fractogram! But I like to think that what I have found is not a bad consolation prize. So, have I discovered something new in mathematics or am I just being irrational?

Hugh Duncan graduated from UCL in 1980 having studied astronomy. He teaches physics and maths in the International School of Nice and is currently writing a popular science book on the topic of fractograms.

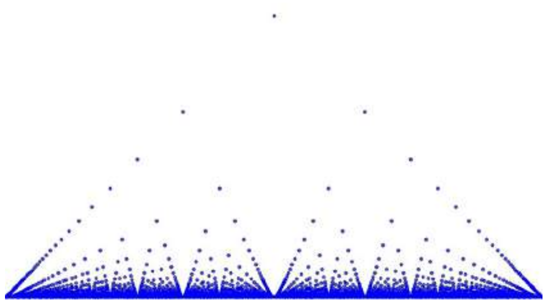
My Favourite Function

The Popcorn Function

Belgin Seymenoğlu

My favourite function is called Thomae's function, but it has many other weird and wonderful names, such as the raindrop function, ruler function, Stars over Babylon and my personal favourite: the popcorn function.

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \text{ is rational and } x = \frac{p}{q} \\ & \qquad \text{(in lowest terms)} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$



What makes the popcorn function remarkable is that it is discontinuous on the rational numbers (or fractions), yet is continuous everywhere else (i.e. on the irrational numbers).

Prize Crossnumber

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#2

Win £100!

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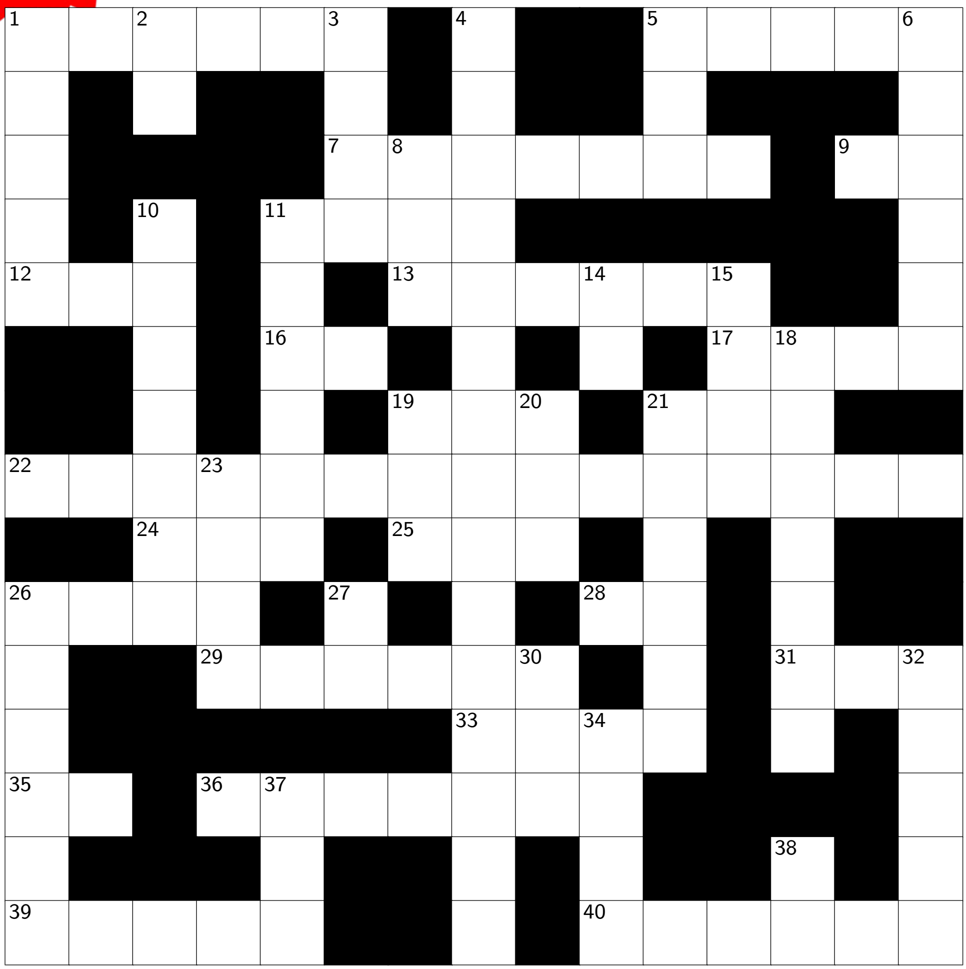


Rules

Although many of the clues have multiple answers, there is only one solution to the completed crossnumber. As usual, no numbers begin with 0. Use of Python, OEIS, Wikipedia, etc. is advised for some of the clues.

To enter, send us the **sum of the across clues** via the form on our website (**chalkdustmagazine.com**) by **5th December 2015**. Only one entry per person will be accepted. Winners will be notified by email and announced on our blog by 19th December 2015. One randomly selected correct answer will win **£100**, and three randomly selected runners up will win a Chalkdust T-shirt. The prizes have been provided by G-Research, researchers of financial markets and investment ideas. Find out more at **gresearch.co.uk**.

Across

- **1** A multiple of 24A. **(6)**
- **5** It is possible to construct a regular polygon with this number of sides using only a ruler and compass. **(5)**
- **7** The number of factors of this number is equal to its fourth root. **(7)**
- **9** A number with 9 proper factors. **(2)**
- **11** The first four digits of 4D. **(4)**
- **12** A prime number. **(3)**
- **13** 30D multiplied by 12A. **(6)**
- **16** The least number of pence which cannot be made using less than 5 coins. **(2)**
- **17** Two less than a triangular number. **(4)**
- **19** The number of consecutive non-prime numbers starting at (and including) 370262. **(3)**
- **21** A prime number. **(3)**
- **22** The smallest number with a (multiplicative) persistence of 11. **(15)**
- **24** The lowest number k such that when 3^k is divided by k the remainder is 24. **(3)**
- **25** When written as a Roman numeral, this number is an anagram of LCD. **(3)**
- **26** A year which began or will begin on a Wednesday. **(4)**
- **28** A multiple of 9. **(2)**
- **29** All the digits of this number are the same. **(6)**
- **31** A square number. **(3)**
- **33** The last four digits of 4D. **(4)**
- **35** The minimum number of knights needed so that each square on a chessboard is either occupied or attacked by a knight. **(2)**
- **36** The number of primes less than 100,000,000. **(7)**
- **39** This number is both square and tetrahedral. **(5)**
- **40** The smallest even number, n, such that $2^n - 2$ is divisible by n. **(6)**

Down

- **1** The sum of the proper factors of 32D. **(5)**
- **2** The sum of this number's digits is 8. **(2)**
- **3** The sum of 34D and 12A. **(4)**
- **4** The 2nd, 4th, 6th, 8th, 10th, 12th and 14th digits of this number are each larger than the digits either side of them. **(15)**
- **5** The sum of this number's digits is 2D. **(3)**
- **6** The sum of 32D, 35A and 1A. **(6)**
- **8** A prime number. **(3)**
- **10** The number of sequences of 16 (strictly) positive numbers such that each number is one more, one less or the same as the previous number and the first and last numbers are either 1 or 2. **(7)**
- **11** A palindrome. **(6)**
- **14** Doubling this number then reversing the digits gives the same results as adding two to this number. **(2)**
- **15** 28A multiplied by the reverse of 5D. **(4)**
- **18** A power of 3. **(7)**
- **19** An abundant number. **(3)**
- **20** The number of degrees Fahrenheit between the boiling and freezing points of water. **(3)**
- **21** All but one of the digits of this number are the same. **(6)**
- **23** 15D plus 17A subtract 34D. **(4)**
- **26** The sum of the digits of this number is 3. **(6)**
- **27** A factor of 25A. **(2)**
- **30** Not a palindrome. **(3)**
- **32** The sum of the proper divisors of 1D. **(5)**
- **34** A square number. **(4)**
- **37** 27D multiplied by 38D. **(3)**
- **38** A multiple of 10. **(2)**

On the cover: *Fermat Point* by Suman Vaze

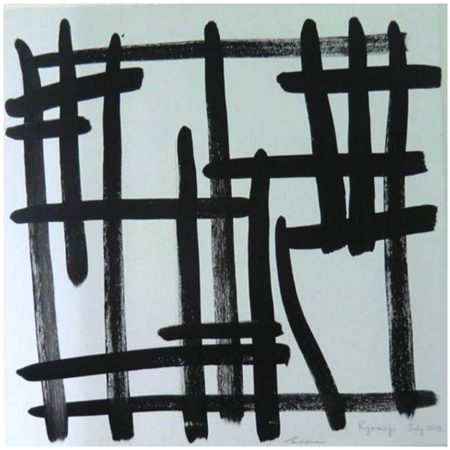
Lady Ponzi by Suman Vaze

Huda Ramli

Suman Vaze sits on her small balcony in crowded, bustling Hong Kong, with a view, just about, of a beautiful Chinese Banyan tree tenaciously growing on a steep stony slope, and paints mathematics. Inspired by the abstract expressionism of Rothko, the radical and influential work of Picasso, and the experimental models of Calder, she fully embodies Hardy's belief that mathematicians are "maker[s] of patterns". Our front cover is one of her pieces: the bold colours proclaim the eponymous *Fermat Point*—the point that minimises the total distance to each vertex of a triangle—along with its geometrical construction. Add an equilateral triangle to each side of the original triangle then draw a line connecting the new vertex of the equilateral triangle to the opposite vertex of the original: the intersection of these lines gives the Fermat point. Not only do these lines all have the same length, but the circumscribed circles of the three equilateral triangles will also intersect at the Fermat point.

Fermat Point by Suman Vaze

All of her paintings combine beauty with a deeper mathematical meaning. Suman describes the rock gardens in Ryoanji Temple in Kyoto as being "enclosed in a rectangular courtyard surrounded by lush Japanese gardens. The rock garden within is an austere arrangement of rocks on neatly raked gravel. Their proportions and positions defy symmetry yet they have an aesthetic balance"



Left: The *Octagonal Numbers* are those of the form $3n^2 - 2n$. Taking the digital roots of the octagonal numbers gives the repeating sequence $1, 8, 3, 4, 2, 6, 7, 5, 9, \ldots$

Right: *The Ryoanji Suite* is constructed in a series of 20 strokes, made so that their intersections formed the consecutive numbers 1 to 9. The artwork was the process of producing the intersections in a particular order and the result of the process is the calligraphy, Ryoanji.

and they moved her to create *The Ryoanji Suite*. *The Octagonal Numbers* also reflect her love for sequences and patterns: "in numbers, shapes, operations, movements, ...".

Much of her inspiration and, one assumes, her motivation comes from the students she teaches in Hong Kong. Through them, she has seen the wide range of emotions that one often experiences when confronted with mathematics: from horror and despair to gratifying relief and triumphant joy. Where a pupil will fall in this scale depends, she believes, on "their approach to the subject and their ability and willingness to immerse themselves in it. Those who enjoy the subject are the ones who are in a sense fearless with new ideas and are able to view it from different perspectives and hence develop greater intuition with the concepts."

For Suman, art provides her with a way to express these concepts. *Lady Ponzi* (on the banner), for example, illustrates the dissection of an equilateral triangle into four pieces that can be reassembled to form a square. The problem was solved in 1902 by Henry Dudeney, an English author and mathematician who set puzzles for various English newspapers and magazines, with the solution having the interesting property that the pieces can be hinged to smoothly rotate between a triangle and a square. Dudeney, incidentally, is also credited for publishing the first crossnumber puzzle.

The wider audience, too, should be grateful for Suman's ability, as her artwork is a window into the beautiful, intricate and magical world of mathematicians, makers of patterns.

*Suman Vaze is an artist and mathematics educator at King George V School, Hong Kong. Her work has been exhibited in mathematics conferences all around the world, including the US, Canada, Hungary, the Netherlands and Korea, along with several solo shows in Hong Kong. For more of her art, go to **sites.google.com/site/vazeart**.*

Huda Ramli is a PhD student at UCL, working on stochastic models of atmospheric dispersion.

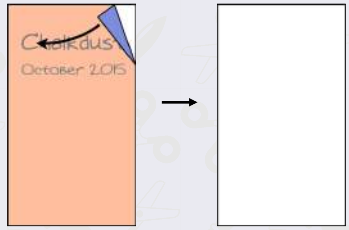
How to Make...
a tube map tetrahedron

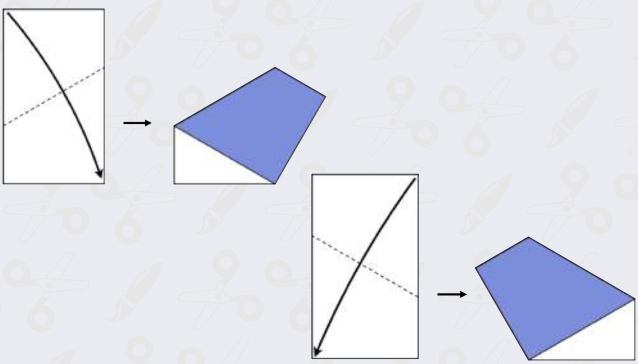
You will need

2 tube maps (or other rectangular leaflets)

Instructions

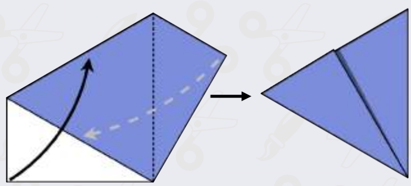
1 Fold over the front cover of each leaflet onto the back. Place the page which you want to appear on the outside of the tetrahedron face down.

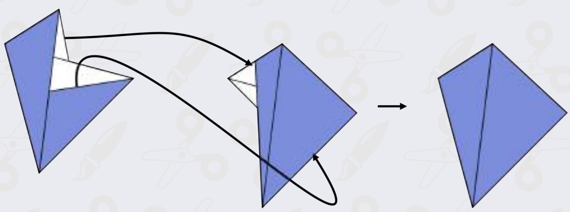




Fold each leaflet diagonally in half, corner to corner. Make sure the second is a mirror image of the first. 2

3 Fold the overhanging ends over. After this step, only parts of the page you want on the faces should be visible.





Open up and fit the two pieces together to make a tetrahedron. Tuck the flaps between the pages for extra strength. 4

You can find more things to make on our website at **chalkdustmagazine.com**

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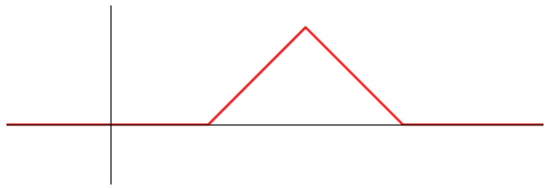
My Favourite Function

Hat Function

Matthew Scroggs

My favourite function is the piecewise linear hat function.

$$f(x) = \begin{cases} 0 & \text{if } x < x_{i-1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x < x_i \\ \frac{x-x_{i+1}}{x_i-x_{i+1}} & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{if } x \geq x_{i+1} \end{cases}$$

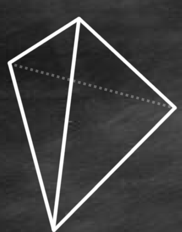


The function is zero outside the range (x_{i-1}, x_{i+1}), one at x_i and linear on the sections (x_{i-1}, x_i) and (x_i, x_{i+1}).

Partial differential equations (PDEs) are a type of equation telling us how various quantities are changing and are used to model a large variety of situations, including those in the fields of acoustics, electromagnetics and quantum mechanics. PDEs are often very hard (or even impossible) to solve, and so numerical methods that give a very good approximation of the solution are required.

One such method is the finite element method, which breaks the x-axis into lots of smaller sections and then uses functions on these sections to make the difficult PDE into a set of simultaneous equations that is easier to solve. The piecewise linear hat function is the most common function used for this method.

Folding Tube Maps



On the opposite page, you will find the instructions for folding a tetrahedron from two tube maps. The resulting tetrahedron is almost (but not quite) regular.

What ratio would the sides of the tube maps have to be to make a regular tetrahedron?

Source: mscroggs.co.uk

Answers at chalkdustmagazine.com/answers

John Forbes Nash
the legacy

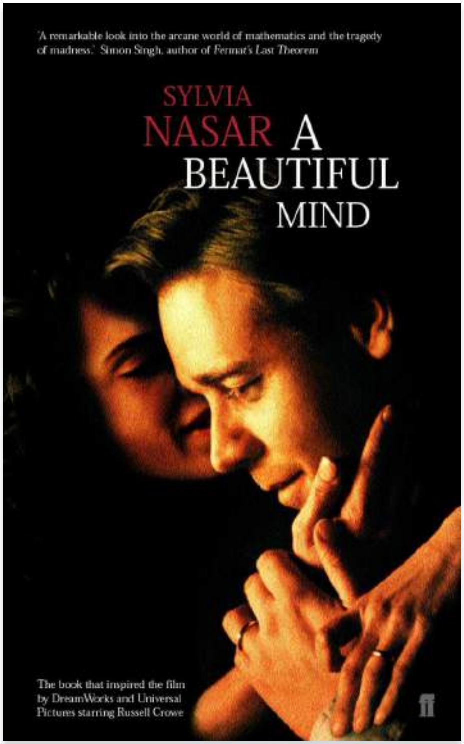
Pietro Servini

When describing John Forbes Nash, Jr (13 June 1928 – 23 May 2015), it's hard to be more succinct than Richard Duffin, a professor at the Carnegie Institute of Technology, who wrote, in his letter of recommendation to Princeton, that "this man is a genius". It was 1948: Nash, having abandoned a degree in Chemical Engineering for one in Mathematics, was only just embarking on a journey that would ultimately make him one of the most famous mathematicians of the 20th Century. Despite the interest of Harvard University, Nash eventually decided to pursue his graduate studies at Princeton and it was there that he published the 317 word paper, *Equilibrium points in N-person games*, that introduced the Nash Equilibrium and won him the Nobel Prize for Economics (jointly with Reinhard Selten and John Harsanyi) in 1994. As a result of this work in game theory, Nash was appointed to the RAND Corporation, which applied this relatively young field to the pressing policy issues of the time: nuclear weapons, the space race, the Cold War.

> *This man is a genius.*

However, Nash's contributions to mathematics go far beyond game theory. Nash was the archetypal problem solver: if there was an important open problem that mathematicians had struggled for years to solve and failed, it warranted his attention—regardless of his possible lack of knowledge in the subject. Indeed, it was often this absence of expertise, coupled with his genius, that would allow him to discover the revolutionary approach required and astound the mathematical community. The papers that became known as the Nash Embedding Theorems, published in

1954 and 1956 and which proved that every Riemannian manifold could be embedded into some Euclidean space, followed the challenge of a fellow faculty member at MIT, Warren Ambrose, who asked "if you're so good, why don't you solve the embedding problem for manifolds?". An unsolved problem in differential geometry that had first been posed by Ludwig Schläfli in 1873, Nash's proof involved the invention of a new technique to solve a system of partial differential equations that had previously been considered unsolvable; a technique now applied in the field of celestial mechanics. The proof is, in the opinion of John Conway, "one of the most important pieces of mathematical analysis in this century"; led Mikhail Gromov to state that "what he has done in geometry is ... incomparably greater than what he has done in economics"; and, ultimately, was partly responsible for Nash's death: on his way back from collecting the Abel Prize for this "striking and seminal contribution to the theory of nonlinear partial differential equations and its applications to geometric analysis", the taxi he was in crashed, killing both him and his wife, Alicia.



Sylvia Nasar's critically acclaimed biography of Nash, *A Beautiful Mind*

Of course, a discussion of Nash's life cannot be complete without a mention of the schizophrenia that robbed the world of his genius for the best part of three decades; and yet, paradoxically, also brought that genius to the attention of the world (in part through Sylvia Nasar's powerful biography, *A Beautiful Mind*, and its Hollywood dramatisation). First hospitalised in 1959, the disease resulted in Nash resigning from his post at MIT that same year, caused his divorce from Alicia in 1962 (they remarried in 2001) and saw him spend much of those lost years wandering the Princeton campus, a phantom. He slowly began to recover and in the 1980s began communicating again with fellow mathematicians, including Harold Kuhn to whom, in a 1996 email, he ascribed his emergence "from irrational thinking" to no medicine "other than the natural hormonal changes of ageing".

In an age where mathematicians became ever more specialised, Nash stood out: his interests and successes ranged from game theory to analysis, geometry to fluid dynamics, nonlinear partial differential equations to cosmology (as a student in Princeton he once approached Einstein to discuss an idea he had had about gravity, friction and radiation). It's pointless to speculate what contributions Nash might have made in these fields were it not for his illness; better to celebrate what Richard Duffin had spotted right at the beginning: his genius.

Nash equilibrium

The mathematical study of decision making can be traced back to Antoine Cournot, a French mathematician, who was the first to publish a theoretical analysis of games in 1838; but it really took off in 1944 with the publication of *Theory of Games and Economic Behaviour* by John von Neumann and Oskar Morgenstern. It became known as game theory, where a game is defined as consisting of players, a set of actions (or strategies) that they can choose from and a pay-off function, which determines what each player receives or loses based on the actions of all the players.

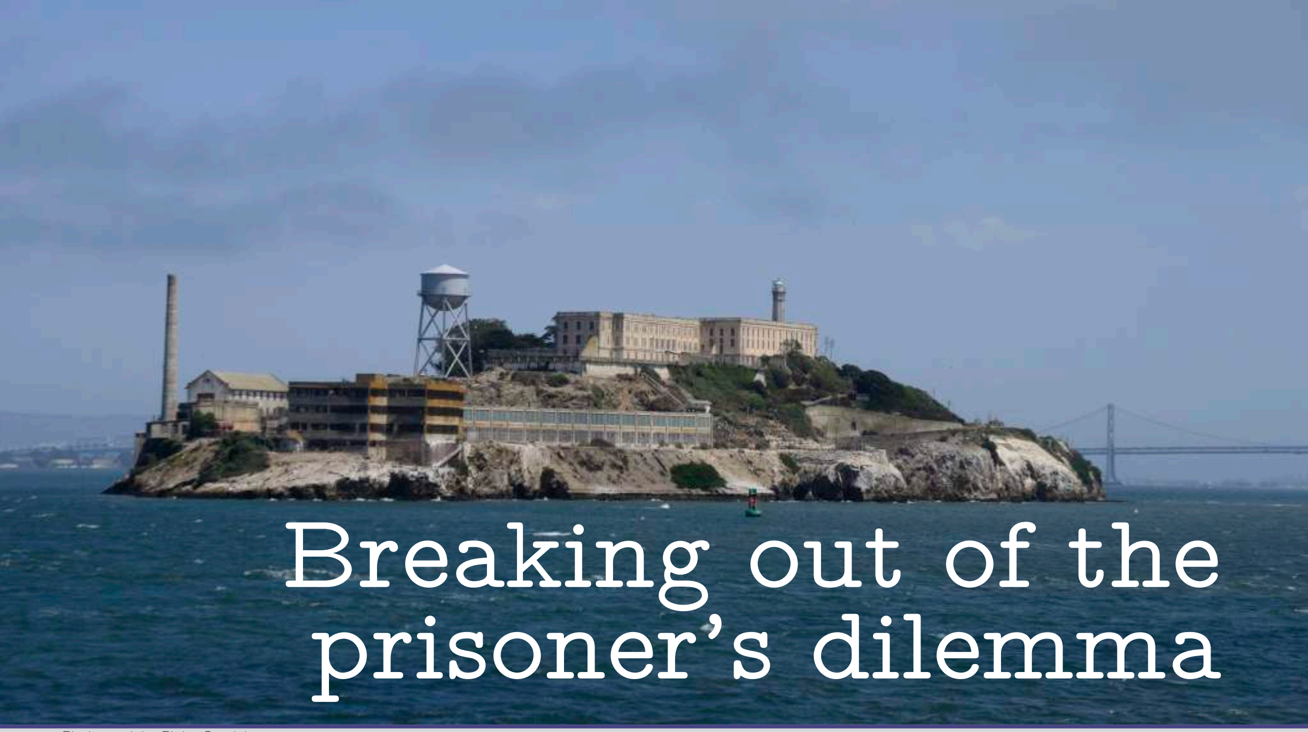
Initially game theory dealt only with zero-sum games: those in which the gains of the players are exactly matched by the losses (in rock-paper-scissors, for example, one person will win and the other will lose). However, although most board games you will play are zero-sum, they have little relevance to real-world issues in economics or, especially important at the time, war. People were beginning to realise that two-person zero-sum games could only be applied to wars of complete extermination; in reality, two opponents will always have some common interest and would have something to gain from cooperation (during the Second World War, for example, the Allies did not destroy the Ruhr's coal mines, knowing that to do so would be counter-productive in the long term).

Fuld Hall, home of the Institute of Advanced Study, where Nash spent much of his time when suffering from schizophrenia.

Nash's 1950 paper (and a later one published in 1951) introduced the Nash equilibrium and revolutionised the approach to game theory, moving it away from zero-sum games. Suppose that there are N players and each player has a strategy, with everyone knowing the strategies of everybody else. Nash equilibrium occurs when nobody can increase their reward by changing strategy. The prisoner's dilemma is often given as an example when explaining Nash equilibrium (see opposite page); another is the stag hunt, where two players can choose whether to hunt a stag or a rabbit. A stag gives more food than a rabbit, but both people need to hunt the stag in order to successfully catch it: if only one does so, then that person will fail and eat nothing. In this game, there are two Nash equilibria: both choose to hunt rabbit and both choose to hunt stag; since if you change your strategy then you get, respectively, either no food or less food. Nash showed that all games would have at least one equilibrium point.

Despite not always being a completely realistic representation of human interactions—we usually don't know what choice everyone else will make, humans often make irrational decisions or mistakes and we might not always trust others to follow their stated strategy (if someone says that they will hunt the stag, are you sure that at the last moment they won't change their mind and go for the rabbit, leaving you to starve?)—game theory and Nash's contribution to it is often used to define policy in war and arms races (as was done when Nash was publishing his papers on the subject), explain social interactions, come up with marketing strategies and develop theories in economics.

Pietro Servini is interested in history and sport. He also happens to be doing a PhD in fluid dynamics at UCL. If he can combine any two of the three it makes him a happy man.



Breaking out of the prisoner's dilemma

Photograph by Pietro Servini

Artiom Fiodorov

The prisoner's dilemma often features in television programmes (such as ITV's Golden Balls) where two contestants have to decide whether they want to share or steal a pot of money. They make their choices in secret from one another and then their decisions are simultaneously revealed.

Let the tuple (a, b) mean that you get a and your opponent gets b: so $(3, -1)$ represents you winning £3 and your opponent losing £1. We introduce the pay-off matrix for a non-iterated (played only once) prisoner's dilemma:

You \ Opponent	Cooperate	Defect
Cooperate	(1, 1)	(-1, 3)
Defect	(3, -1)	(0, 0)

Each player is given the opportunity either to defect or to cooperate. In the original set-up with prisoners the option of defecting meant betraying the other and testifying whilst cooperation represented remaining silent.

Nash equilibrium and Pareto efficiency

The definition of *Nash equilibrium* (NE) implies that if all but one players are playing at the Nash equilibrium strategy, the other player is better off playing it too. So in our pay-off matrix above, (Defect, Defect) is a Nash equilibrium strategy since if you know that your opponent defects, you should defect too. (Cooperate, Cooperate) is not NE since if you know that your opponent cooperates, you should defect.

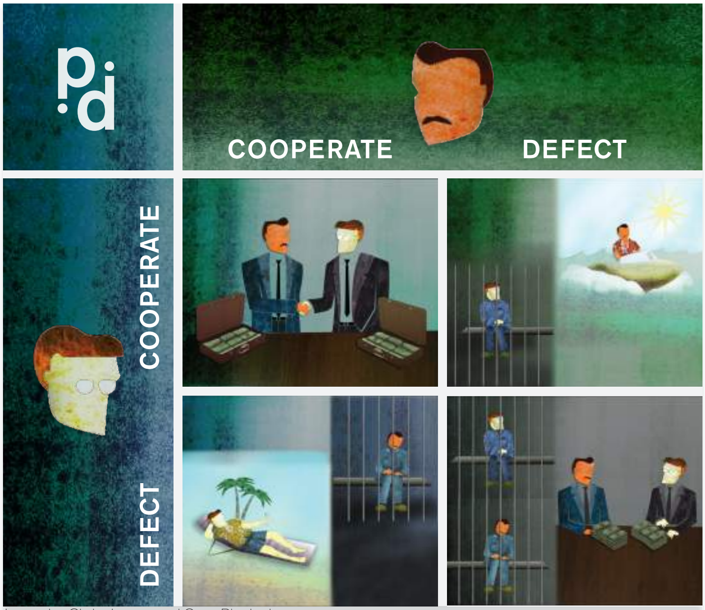


Image by Chris Jensen and Greg Riestenberg

The prisoner's dilemma

It is often taught in game theory courses that two rational players should decide to defect as each one is better off defecting no matter what the other player does. This yields to the paradoxical situation of the players foregoing a more favourable outcome to both.

Pareto efficiency is defined as a state where it is impossible to make any one player better off without making at least one contestant worse off. In our example it is clear that (Cooperate, Cooperate) is Pareto efficient, but the (Defect, Defect) outcome is not. This leads to one criticism of Nash equilibrium: the outcome of NE isn't always Pareto efficient.

Many possible remedies for a more satisfactory solution of the prisoner's dilemma have been advocated. It is common to play the game repeatedly to show that rational players cooperate on the first few rounds but then start defecting.

A colleague and I stumbled upon the prisoner's dilemma whilst discussing a different problem and we originally had a mild disagreement over what the optimal strategy should be for our scenario. However, after a short conversation we settled it. Here is how we did it.

Let the probability guide us

Assume that you (P1) believe that the opponent (P2) cooperates with probability p_2. Your task is to find the probability of cooperating p_1, such that the expected pay-off is maximised. I would like to emphasise that p_2 represents your confidence in the statement that P2 will defect. p_2 is *not* the proportion of times P2 defects if you played the prisoner's dilemma many times. This interpretation, also known as *Frequentist*, would be problematic for a game played only once. Instead, p_2 is simply your state of knowledge about P2. Such a *Bayesian* interpretation will protect us against mind projection fallacy: the fallacy when we confuse our state of knowledge about reality with the reality itself. For example, if we raise p_2 to 1 does that mean the opponent will cooperate? No, we just *think* that he will. This will not, telepathically, cause him to cooperate.

Short foray into plausibility reasoning: if $A \implies B$ and you observe B, does that make A more plausible? Bayes' theorem says yes. This aspect is ignored in classical game theory, but we will rely on it here.

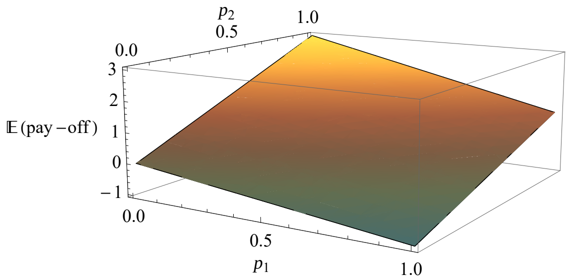
Finding the Nash equilibrium

The strategy assumes that the two players make their decisions independently: your choice of p_1 does not affect what you believe p_2 to be. Hence the expected pay-off is found to be

$$\mathbb{E}[\text{pay-off}] = p_1p_2 - p_1(1-p_2) + 3(1-p_1)p_2,$$

which reduces to

$$\mathbb{E}[\text{pay-off}] = 3p_2 - p_1(1+p_2).$$



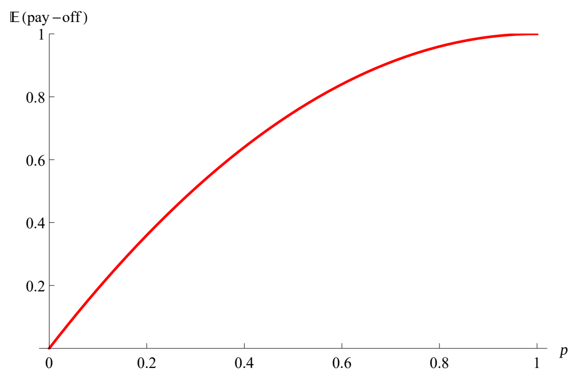
Expected pay-off when p_1 is independent of p_2

The maximum expected pay-off is always achieved at $p_1 = 0$ for any value of p_2 between 0 and 1. So you never cooperate. The probability set-up is now redundant as your strategy is deterministic. Such a deterministic strategy is also known as pure. We arrived at the NE strategy again: a strategy which is often referred to as rational, and yet I'd hesitate to refer to people who leave the best option on the table, $(1, 1)$, as rational.

What if you played it with a clone of yourself?

Yeah, that's right. Let's just clone you for the purpose of the game just before it starts and see how this goes. Because why not? Previously p_2 was independent of p_1, but can we now say that your choice of p_1 doesn't change your belief about p_2? The set-up now is as symmetric as it can possibly get and if you think you are going to cooperate with probability p_1, is there any reason to suppose that your clone cooperates with $p_2 \neq p_1$? Well... I argue that there isn't. So let's say that $p_1 = p_2 = p$. Therefore

$$\mathbb{E}[\text{pay-off}] = p^2 - p(1-p) + 3(1-p)p = -p(p-2)$$

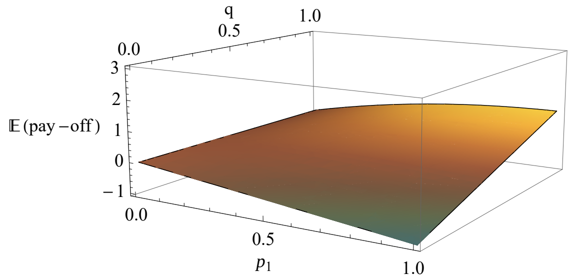


Expected pay-off when you play against a clone of yourself

So we have an inverted parabola with roots at 0 and 2 readily maximised at $p = 1$. So you always cooperate with your own clone. The strategy is, again, deterministic. The intuition is also satisfied: if I defect then the clone defects because of the same reasoning and we both get nothing. If I cooperate then my clone cooperates and we are both happy. Seems logical that we both choose to cooperate.

Back to reality

What if your opponent dresses like you, talks like you, acts like you, but is not quite you? Let us quantify the "not quite you" part as follows: the person you are playing with flips a coin in the morning. If it comes up heads he will defect. It's just one of those days when he's angry at everyone. He acts just like you would act otherwise. Say the coin has probability q of coming up tails.



Expected pay-off when your opponent is not quite you

$$p_2 = \begin{cases} 0, & \text{with probability } 1-q \\ p_1, & \text{with probability } q. \end{cases}$$

We call q the similarity index: it tells you how similar you think your opponent is to you. Hence, ignoring the details, $\mathbb{E}[p_2] = qp_1$ and

$$\mathbb{E}[\text{pay-off}] = p_1(3q-1)\left(1 - \frac{q}{3q-1}p_1\right),$$

which is maximised at

$$p_{\text{max}} = \begin{cases} 0, & \text{if } q \leq \frac{1}{3} \\ \frac{3q-1}{2q}, & \text{if } q > \frac{1}{3} \end{cases},$$

where p_{max} is dependant on the pay-off matrix. Note that the strategy isn't deterministic for $1/3 < q < 1$ and it sits right in-between Nash equilibrium and the one with a clone. Our intuition is again satisfied: the chance of you cooperating is determined by the pay-off matrix and by the similarity of the opponent's thinking process to yours.

I cannot emphasise enough that even though it seems like P2 telepathically knows about your strategy, there is no telepathy or magic thinking involved. The qp strategy is what *you believe* P2 will do and it has nothing to do with what he will actually do.

Putting it in perspective

Should your belief about p_2 depend on your strategy p_1? I don't see why it shouldn't. Biological neural networks in our heads run on the same principles in all of us. So if I convince myself that I should defect then I have anthropomorphic evidence staring right at me that the opponent might think to do the same thing. Therefore I should revise my belief about p_2. Another way to put it: we are all "imperfect" clones of each other.

Let us set up a chain of propositions that could fit this. Let A(X) = "X proportion of people choose to defect", B = "I end up defecting" and C = "My opponent ends up defecting". Now observing B could raise your estimate of X, which in turn would raise your confidence in C. Note that there is *no* physical causality involved: it would be preposterous to reason that your decision causes your contestant to change his mind. However, there is a logical causality that governs the probability flow in your mind.

Only when you can verifiably be sure that you are playing against a P2 whose strategy is completely independent from yours, for example if you are playing against rolls of a die, should you stick to the Nash equilibrium strategy.

Different pay-off matrices

We mentioned that p_{max} was dependent on the pay-off matrix, so let us play around with our pay-off matrices to check how our solution performs. Keep (Cooperate, Cooperate) and (Defect, Defect) fixed, but introduce (b, c) instead for (Cooperate, Defect), such that $b > 1$ and $c < 0$.

You \ Opponent	Cooperate	Defect
Cooperate	(1, 1)	(c, b)
Defect	(b, c)	(0, 0)

It can now be shown that

$$p_{\text{max}} = \begin{cases} 0, & \text{if } q \leq -\frac{c}{b} \\ \frac{bq+c}{2(b+c-1)q}, & \text{if } q > -\frac{c}{b} \end{cases}$$

under a technical assumption that $b + c \geq 2$.

You can see that $p_{\text{max}} \to 1/2$ as $b \to \infty$ for any fixed q, c. Makes sense? You and your contestant are aiming for the b outcome, but only one of you can have it. Might as well let randomness decide who is going to get it. And if $c \to -\infty$ you should stop cooperating.

Connection with a cooperative game theory

It is apparent that when one plays against a clone it is equivalent to deciding upon the strategy beforehand (at time 0) and sticking to it throughout (without the ability to communicate any longer). The field of game theory where players are permitted to decide upon a strategy first and are obliged to stick to it is also known as *cooperative game theory*. In a cooperative prisoner's dilemma players can easily achieve the Pareto efficient outcome (C, C) as long as there is a third party ready to enforce the contract.

In fact, our "back to reality" scenario is equivalent to playing such a cooperative dilemma only with a person who is somewhat unreliable and feels the same way about you: then you can both pre-agree that maximising $p(3q-1)(1-qp/(3q-1))$ is the right thing to do.

So it turns out that allowing plausibility reasoning about your opponent's strategy based on your own strategy in a classical game theory setting, as we have done here, leads us to a cooperative game theory. It appears that it is beneficial for everyone to act as if they signed an implicit contract

Using cooperative game theory to explain social phenomena such as queueing.

rather than out of a selfish self-interest. Perhaps this could explain a number of social concepts like queueing, voting, not littering, etc.

What if the game isn't symmetric?

The solution to these kind of dilemmas now easily follows. Imagine that at time 0 you could pre-agree with the other players about a strategy, what would it be? Just pick the one that leads to a Pareto efficient outcome and stick to it. Hence if you play with clones that's what you should do. We believe that Pareto efficiency implies that the strategies $s_1^*, s_2^*, \ldots, s_N^*$ maximise

$$\min_{1 \le n \le N} \mathbb{E}[\text{pay-off for player } n]$$

because no player would agree to reduce his expectation for the gain of another player.

Let's work through the "back to reality example", but with different trust issues. At time 0, P1 and P2 meet and disclose:

P1: "Let us make an agreement that you cooperate with chance p_2, but I estimate $(1 - q_1)$ chance that you will break the agreement and defect regardless."

P2: "Similarly, I think there is a $(1 - q_2)$ chance of you violating your agreement of cooperating with probability p_1."

So, with the same pay-off matrix as before, the expected pay-off for player 1 is

$$F_1(p_1, p_2) = -p_1 + 3p_2 - p_1 p_2$$

and the pay-off for player 2, $F_2(p_1, p_2)$, is simply $F_1(p_2, p_1)$. For each choice of (q_1, q_2), the formula for what values of (p_1, p_2) we achieve the Pareto efficient outcome can be derived to obtain

$$(p_1^*, p_2^*) = \text{argmax}_{p_1, p_2} \min\left[F_1(p_1, q_1 p_2), F_1(p_2, q_2 p_1)\right]$$

By symmetry, for $q_1 = q_2 = q$ this is achieved at $p_2^* = qp_1^*$. For the asymmetric case solving it numerically yields the following values:

q_1, q_2	0.75, 1	0.5, 1	0.25, 1	0.1, 1	0.75, 0.75	0.5, 0.75	0.25, 0.75
p_1^*	0.86	0.66	0.33	0	0.83	0.61	0.25
p_2^*	0.99	0.93	0.67	0	0.83	0.75	0.44

One interesting choice of (q_1, q_2) is $(0.1, 1)$: when the opponent trusts you unconditionally, but knows you don't trust him. In that case you are both better off defecting.

In the end, using this method we can find the solution that takes into account how similar you think you appear to your opponent and how similar you think your opponent is to you!

Obligatory references and other titbits

Obviously it turned out that what Stephen and I stumbled upon one evening isn't new. Douglas Hofstadter introduced the concept of *superrationality* which is equivalent to playing against yourselves. However, to the best of my knowledge, this interpretation of rationality is still not widely accepted in the orthodox literature.

And finally, some relevant facts from Golden Balls for your information:

- Individual players on average choose "split" 53% of the time.
- There is little evidence that contestants' propensity to cooperate depends positively on the likelihood that their opponent will cooperate (i.e. little evidence for conditional cooperation).
- Contestants are less likely to cooperate if their opponent has tried to vote them off the show in the first two rounds of the game, which is in line with the notion that people have an intrinsic preference for reciprocity.

*Artiom Fiodorov is a PhD student at UCL who (evidently) likes writing. Cognitive science, maths and programming are his recurring themes. When he is not writing for Chalkdust or his blogs (**afiodorov.github.io**) he is studying random walks in random environments.*

Many thanks to Stephen Muirhead for solving the asymmetric case and plenty of other suggestions.

My Favourite Function

Function Function

Matthew Wright

Let $\mathbb{S}$ be the set of all strings of letters from the Roman alphabet. Then my favourite function is the map $F : \mathbb{S} \to \mathbb{S}$ such that

$$F(x) = \begin{cases} \text{fun}\,x & \text{if } x = \text{ctions} \\ x & \text{otherwise} \end{cases}$$

Why do I like this function? Because it puts the fun in functions! (sorry...)

¡TOP TEN!

This issue, Top Ten features the best mathematical symbols! To vote on the Top Ten models of calculator go to **chalkdustmagazine.com**

10 — y

A new entry at 10, everyone's least favourite axis... y!

9 — $\neq$

At 9, allegedly resembling the British Rail logo... the not equals symbol!

8 — $\perp$

The perpendicular symbol gets its angle just right this week at number 8!

7 — $\ll$

A re-entry at 7: The Bee Gees with Much Less Than a Woman.

6 — ε

It may be small, but we love it: epsilon spends yet another week at number 6.

5 — $\propto$

At 5, amid protests of plagiarism from alpha: the proportionality symbol.

4 — $\in$

Down two places to 4, showing it really belongs in the top ten, it's... ∈!

3 — $\lfloor\ \rfloor$

Maybe its resemblance to sideburns makes it cool: the floor symbol is at 3.

2 — x

At 2, we've been searching for years and we still haven't found it: it's x!

1 — $\forall$

And topping the charts, it's a new entry. The perfect symbol to represent some upside down screaming: for all!

∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀∀

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A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

G. H. Hardy, 1877–1947