A dummy first page

If you want to print 2-up, run of from page 2 (the next page).
This will get the book page number in the correct corner.

A point-free and point-sensitive analysis of the patch assembly

Rosemary A. Sexton

Mathematical Foundations Group
The University
Manchester

This is a version of the PhD Thesis Submitted to the then Victoria University of Manchester in September 2003.

This version doesn't use the silly spacing and formatting required by the University regulations.

Contents

1 Introduction **3**

2 Point-sensitive background **7**
- 2.1 Basic separation properties, sobriety and regularity . . . 7
- 2.2 Saturation and compact saturated sets . . . 10
- 2.3 The front topology . . . 13

3 Point-free background **17**
- 3.1 Frames . . . 17
- 3.2 Filters . . . 20
- 3.3 The point space of a frame . . . 21
- 3.4 The Hofmann-Mislove Theorem . . . 25
- 3.5 The frame separation properties regular and fit . . . 27

4 The point-sensitive patch construction **31**
- 4.1 Packed spaces . . . 31
- 4.2 The point-sensitive patch construction . . . 32
- 4.3 The point-sensitive patch is not always sober . . . 33
- 4.4 Functorial properties of the point-sensitive patch construction . . . 35

5 The full assembly **39**
- 5.1 Nuclei and related gadgets . . . 39
- 5.2 The u, v and w nuclei . . . 46
- 5.3 Spatially induced nuclei . . . 50
- 5.4 The Cantor-Bendixson example . . . 53
- 5.5 Admissible filters and fitted nuclei . . . 55
- 5.6 Nuclei associated with open filters . . . 59
- 5.7 Block structure . . . 60

6 Properties of the full assembly **65**
- 6.1 The full assembly is a frame . . . 65
- 6.2 N is a functor . . . 66
- 6.3 The fundamental triangle of a space . . . 70
- 6.4 The point space of the full assembly . . . 73

7 The patch assembly **77**
- 7.1 The construction . . . 77
- 7.2 Functorality matters . . . 79

7.3 The full patch assembly diagram 85

8 A hierarchy of separation properties 87
8.1 Patch triviality 87
8.2 Stratified tidiness 89
8.3 Stratified regularity 91
8.4 The spatial case 93
8.5 Stacked spaces 94
8.6 Vietoris points 100

9 The point space of the patch assembly 103
9.1 The basic information 103
9.2 Two spoilers 104
9.3 The 'ordinary' points of the patch assembly 105
9.4 The wild points of the patch assembly 107

10 Examples 111
10.1 The cofinite and cocountable topologies 111
10.2 A subregular topology on the reals 122
10.3 The maximal compact topology 122
10.4 A glueing construction 125

11 The boss topology on a tree 129
11.1 Trees and boss topologies 129
11.2 Stacking properties of a boss space 134
11.3 Full splitting trees 139
11.4 Some wide trees 142
11.5 The top down tree 145

Chapter 1

Introduction

Point set topology has a history of somewhat over 100 years, and has been used in many different areas of mathematics as well as in other applications. However, many of these uses are in the broad area of continuous mathematics, as opposed to discrete mathematics. In consequence, the topological spaces which have received the most attention are T_2 (Hausdorff), whereas many of the spaces of interest to theoretical computer scientists are usually not so well separated. Here we will study spaces which may have weaker separation properties.

Each topological space carries a specialisation order which, in general, is a pre-order. The space is T_0 precisely when this comparison is a partial order, and is T_1 when it is equality. The pre-order does not determine the topology, although there are some canonically associated examples.

In the standard way we have the short hierarchy

$$T_0 \qquad T_1 \qquad T_2 \qquad T_3$$

of separation properties, where T_3 is the property T_2 strengthened by the addition of regularity. These are the bread and butter of point set topology. Here we will take a look at what can happen in between these separation properties.

This outlines an argument for a study of the category Top of topological spaces and continuous maps, with an emphasis on those spaces with separation properties lying between T_0 and T_3. However, there is another observation which should be taken into account.

A topological space is a set of points structured by the carried topology. Often in applications some of these points can be viewed as concrete entities, but others are idealised and are there merely to complete the structure. Furthermore, often the individual points are not of great interest, as we are concerned more with the way certain collections of points (the open sets) interact. Given this, it makes sense to concentrate on the algebra of open sets rather than the space of points. In doing this, we move from the environment of point-sensitive topology to that of point-free topology. This idea has been pursued in mathematics for almost 30 years, and is quite common within computer science. A somewhat eccentric account is given in [18]; a more balanced view can be found in [9]. Although the latter was written purely from a mathematical perspective, it has had a great deal of influence on the development of this area of theoretical computer science.

Just as the category Top is the appropriate environment for point-sensitive topology, there is a category Frm which is the appropriate environment for point-free topology.

These two categories are intimately connected but are by no means equivalent. The category Frm is much more flexible than Top and contains many more facilities, some of which couldn't be imagined by staying within Top. Much of my research exploits this greater flexibility.

We will develop the interplay between Top and Frm. In particular, we often use a point-free analysis, done within Frm, to obtain point-sensitive information about Top. This is a fruitful approach since, as mentioned earlier, the category Frm provides many facilities not easily obtainable within Top.

From an historical perspective the starting point for this research was the paper by Escardó [4] on stably locally compact spaces. These spaces often appear in connection with domains, as do the more general locally compact spaces. These spaces can be studied from both a point-sensitive and a point-free angle; the difference is more a matter of taste than content. However, there is one construction that is used quite a lot for these and more general spaces, and which seems to be quite firmly set in the point-sensitive environment. This is the associated patch space of a space. Escardó showed that for this restricted class of spaces the patch topology can be produced in a purely point-free fashion and, furthermore, this construction has some nice functorial properties.

An analysis of part of [4] was carried out in [13] as an MSc project. It became clear that Escardó's results depend quite heavily on the coincidence of two techniques: the use of Scott continuous nuclei, and the use of certain fitted nuclei. In general, these are quite separate gadgets, but for the setting in which Escardó worked they are intimately connected and his results depend on this connection.

On the positive side, it also became clear that a version of the patch construction could be developed in a purely point-free fashion and which could be used throughout Frm, not just on a restricted family of objects. However, this raised more questions than answers, and it was evident that a deeper analysis was needed. This is what I have attempted.

It is not useful to give the chronology of the various results obtained (for, as often happens, they came in quite a random order). However, a quick look at the contents of each chapter will help. In the survey some technical words will be used without explanation. Of course, these will be defined at the appropriate point in the body of the thesis.

Chapter 2 gathers together the background point-sensitive material, that is the required standard results from point set topology concerning the category Top. There is nothing new in this chapter, although perhaps Corollary 2.3.6 has some novelty value.

Chapter 3 organises the required background point-free material concerning the category Frm. Again there isn't much that is new here, but the account of the Hofmann-Mislove result (in Section 3.4) contains information that doesn't seem to be as well known as it should be.

As indicated above, many areas of mathematics use only T_2 spaces, but not all spaces have this property. Thus there is a question of how a 'defective' space S may be 'corrected' to obtain a T_2 space. The topic of **Chapter 4**, the point-sensitive patch construction, can be viewed as an attempted 'correction'. In each T_2 space each compact saturated set is closed. The point-sensitive patch construction attempts to achieve this by simply declaring that each compact saturated set should be closed. This chapter gathers together the relevant properties of the construction. There are some new results here, or at least results which do not seem to be well known. In particular, we isolate the classes of *packed* and *tightly packed* spaces.

The category Frm offers more facilities than the category Top. One of these is the full assembly of a frame. (This exists for a topology but, in general, is not itself a topology.) The assembly NA of a frame A is a gadget which has a great influence on the structure of A. It is a vehicle which enables many calculations to be carried out. **Chapter 5** deals with nuclei, the building blocks of the assembly. Much of the material is standard, but there are some new aspects. In particular we show how each *open filter* on a frame induces a chain of elements ascending through the frame. The length of such chains form a measure of the complexity of the frame. We also take a look at the *block structure* of an assembly.

Chapter 6 continues the analysis of the full assembly. The previous chapter concentrated on the elementary algebraic properties whereas this chapter concentrates on the functorial properties. In particular, although not entirely new, the results of Sections 6.3 and 6.4 provide useful information.

Chapter 7 is where the thesis begins to break new ground. In Chapter 4 we attached to each space S a patch space ${}^{p}S$ and a frame embedding

$$\mathcal{O}S \hookrightarrow \mathcal{O}^{p}S$$

between the topologies. In Chapters 5 and 6 we attached to each frame A a frame NA, its assembly, and a frame embedding

$$A \hookrightarrow NA$$

(up to isomorphism). We are now in a position to mimic the point-sensitive construction

$$S \longmapsto {}^{p}S$$

to attach to each frame A a patch assembly PA which sits between A and its full assembly NA.

$$A \longrightarrow PA \hookrightarrow NA$$

With a bit of hand waving it could be argued that the construction PA on a frame A is the point-free analogue of the point-sensitive construction ${}^{p}S$ on a space S. However, as we will see, this is not entirely correct. This chapter sets up all the basic properties of $P(\cdot)$. By the end of the chapter we have attached to each space S a surjective frame morphism

$$P\mathcal{O}S \xrightarrow{\pi_S} \mathcal{O}^{p}S$$

from the patch assembly of its topology $\mathcal{O}S$ to the topology of its patch space ${}^{p}S$. The obvious inclination is to think that π_S is an isomorphism. And it is for nice spaces, but not in general. Thus we need to investigate further.

We have attached to each frame A an embedding

$$A \longrightarrow PA$$

into its patch assembly. When is this an isomorphism? A standard result shows that if A is regular (as for the topology of a T_3 space) then we have an isomorphism. A refinement of this shows that we still have an isomorphism when A is the topology of a T_2 space. **Chapter 8** contains a thorough analysis of when this embedding is an isomorphism. Much of the analysis is done in a point-free setting, but for the purposes of this review let's consider only the point-sensitive results. Eventually we show that for the topology $A = \mathcal{O}S$ of a T_0 space S the embedding is an isomorphism precisely when S is

$$T_1 + \text{ sober } + \text{ packed } + \text{ stacked}$$

where the packed property is discussed in Chapter 4 and the stacked property is introduced in this chapter. (It turns out that T_1 and sober are both implied by the other two properties.) We find there are two interlacing hierarchies of separation properties descending from T_3 down to this composite property. These are

$$\alpha\text{-regular} \qquad \alpha\text{-tidy}$$

where α is an ordinal. For a T_0 space we have

$$0\text{-regular } = T_3 \qquad 1\text{-tidy } = T_2$$

and

$$\alpha\text{-tidy} \implies \alpha\text{-regular} \implies (\alpha+1)\text{-tidy}$$

for each ordinal α. This measure α is related to the length of certain chains introduced in Chapter 5. Later we show, by example, that this hierarchy does not collapse. Finally, in this chapter, we observe an unexpected connection with the Vietoris points of a space.

In Chapter 7 we saw that for each sober space S there is a surjective frame morphism

$$P\mathcal{O}S \xrightarrow{\pi_S} \mathcal{O}^p S$$

connecting the two patch constructions. In general this need not be an isomorphism. What is the connection between ${}^pS$ and the point space of $P\mathcal{O}S$? More generally, for a frame A with point space S, what is the connection between ${}^pS$ and the point space $\mathsf{pt}(PA)$ of the patch assembly? We find there is a canonical embedding

$${}^p\mathsf{pt}(A) \longrightarrow \mathsf{pt}(PA)$$

which, under certain reasonably nice circumstances, is a homeomorphism. However, for some frames A (even spatial frames) the assembly PA can have 'wild' points, that is points that are not visible in $\mathsf{pt}A$. **Chapter 9** begins an analysis of these points. However, as often happens, our results seem to generate more questions than answers.

Much of this research has been concerned with finding counterexamples to various questions that have arisen. These examples are referred to within the thesis and often the same example is used for several purposes. Rather than scatter the details throughout the thesis, we have gathered the examples together in the final two chapters.

Chapter 10 collects together a variety of examples some of which have been found in the literature and some of which have been purpose built.

Finally, in **Chapter 11** we describe a large family of examples within a common format. Each space is a tree $\mathbb{S}$ with an added point $*$, the boss point, which is used to control a topology on the space $S = \mathbb{S} \cup \{*\}$. The methods used to analyse S seem quite general and perhaps could be extended to cover other trees.

Chapter 2

Point-sensitive background

In this short chapter we collect together various facts from point-set topology. There is nothing new here (but one result is not as well known as it should be). The chapter helps to fix various notations and conventions.

Let S be a topological space. We write

$$\mathcal{O}S \qquad \mathcal{C}S$$

for, respectively, the family of open sets (the topology) and the family of closed sets of S. For an arbitrary subset $E \subseteq S$ we write

$$E^{\circ} \qquad E^{-} \qquad E'$$

for, respectively, the interior, the closure, and the set theoretic complement of E in S.

We tend to use

$$U, V, W, \ldots \qquad X, Y, Z, \ldots \qquad E, F, G \ldots$$

to range over

open closed arbitrary

subsets of S respectively. However, there will be times when this informal convention is broken.

We often say 'space' as an abbreviation of 'topological space' (for no other kind of space occurs here).

2.1 Basic separation properties, sobriety and regularity

We will make frequent use of the standard topological separation properties T_0, T_1 and T_2.

We also use the topological notion of sobriety.

Definition 2.1.1. A non-empty closed set X is *irreducible* in a space S if

$$\left.\begin{array}{l} U \text{ meets } X \\ V \text{ meets } X \end{array}\right\} \Longrightarrow U \cap V \text{ meets } X$$

for every two open sets U and V.

A space is *sober* if it is T_0 and every closed irreducible set is a point closure. □

This notion of sobriety is equivalent to the requirement that each closed irreducible set is a unique point closure. Uniqueness is given by the T_0 property.

Most of the spaces we consider are sober, but a few are not. One pair of non sober spaces that crop up occasionally are the cofinite and cocountable topologies of Section 10.1.

Every T_2 space is sober, but the properties T_1 and sobriety are incomparable; we can have either one without the other.

If a space is not sober, there is a way to sober it up.

Definition 2.1.2. Let S be a topological space. The *sober reflection* of S (written ${}^{+}S$) is the topological space whose points are the closed irreducible sets of S. For each $U \in \mathcal{O}S$ let ${}^{+}U \subseteq {}^{+}S$ be given by

$$X \in {}^{+}U \iff U \text{ meets } X$$

for each $X \in {}^{+}S$. Then

$$\mathcal{O}^{+}S = \{{}^{+}U \mid U \in \mathcal{O}S\}$$

is the appropriate topology on ${}^{+}S$. □

It is straightforward to check that $\mathcal{O}^{+}S$ is a topology on ${}^{+}S$ and later that

$$\begin{array}{rcl} \mathcal{O}S & \longrightarrow & \mathcal{O}^{+}S \\ U & \longmapsto & {}^{+}U \end{array}$$

is a frame morphism in the sense defined in 3.1.4.

Furthermore, ${}^{+}S$ is sober as expected and the assignment

$$\begin{array}{rcl} S & \longrightarrow & {}^{+}S \\ p & \longmapsto & p^{-} \end{array}$$

is continuous. The justification for calling this construction the sober *reflection* is provided by the following lemma which we state without proof.

Lemma 2.1.3. *For each continuous map*

$$S \xrightarrow{\ \phi\ } T$$

from an arbitrary space S to a sober space T, there is a unique continuous map

$${}^{+}S \xrightarrow{\ {}^{+}\phi\ } T$$

such that

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & T \\ & \searrow \quad \nearrow_{{}^{+}\phi} & \\ & {}^{+}S & \end{array}$$

commutes.

It is possible to produce the sober reflection in a neater way by using the point space of a frame, as we will see in Lemma 3.3.10.

When we construct the sober reflection of a particular space S, the notation can become unnecessarily complicated if we insist on using the closed irreducible sets themselves to label the points of ${}^{+}S$. Instead we will often find it convenient to use the original points of S (to stand for their point closures) and add in extra points as representatives of the closed irreducible sets that are not already point closures. This is what we do when we look at the cofinite and cocountable topologies in Section 10.1; we throw in an extra point with the necessary properties so that the whole space becomes a point closure.

On occasion we will need a space which is T_1 and sober but not T_2. It seems plausible that such a space could be constructed as the sober reflection of a T_1 space. It might seem plausible, but it never works.

Lemma 2.1.4. *If a space S has a sober reflection ${}^{+}S$ which is T_1, then S is already sober and T_1.*

Proof. We know that $S \subseteq {}^{+}S$ where S carries the subspace topology. Suppose $X \subseteq S$ is a closed irreducible set of S, and consider the closure X^{-} of X in ${}^{+}S$. If $U \in \mathcal{O}{}^{+}S$, we have

$$X^{-} \text{ meets } U \Longrightarrow X \text{ meets } U \Longrightarrow X \text{ meets } S \cap U$$

and hence X^{-} is irreducible in ${}^{+}S$. Now, ${}^{+}S$ is T_1 and sober, and so $X^{-} = \{p\}$ for some $p \in {}^{+}S$, and hence $X = \{p\}$. □

A space is T_1 and sober precisely when each closed irreducible set is a singleton. As we will see later, spaces that are both T_1 and sober but not T_2 can be tricky to find.

The separation property regularity is commonly used to analyse topological spaces. It is a strong condition that makes many of the constructions we are studying trivial, so most of the spaces we study are not regular. In Section 3.5 we will extend the concept of regularity to a point-free setting, and later on we weaken this to give a hierarchy of separation properties connected with the patch assembly.

Definition 2.1.5. A topological space S is *regular* if for each $p \notin X \in \mathcal{C}S$ there are $U, V \in \mathcal{O}S$ such that

$$p \in U \qquad X \subseteq V \qquad U \cap V = \varnothing$$

hold. □

A space can be regular without being T_0, and for this reason we make use of the property

$$T_3 = T_0 + \text{ regular}$$

which fits into the usual hierarchy

$$T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

in a natural way.

2.2 Saturation and compact saturated sets

Every topological space carries a pre-order on its points. This is the specialisation order and it gives us some useful information about the structure of the space.

Definition 2.2.1. Let S be a topological space. The *specialisation order* on S is the comparison $\sqsubseteq$ given by

$$p \sqsubseteq q \Longleftrightarrow p^- \subseteq q^-$$

for $p, q \in S$. □

It is almost trivial that this comparison $\sqsubseteq$ is a pre-order (that is, it is reflexive and transitive). It is a partial order precisely when the space S is T_0, and it is equality precisely when S is T_1.

In this thesis we are concerned almost exclusively with spaces that are at least T_0, so we consider the specialisation order to be a partial order.

Using this partial order on a T_0 space, we can introduce the concept of saturation. In simple terms, a set is saturated when it is an upper section with respect to the specialisation order.

Definition 2.2.2. Let $(S, \leq)$ be a poset. For each subset E of S

$$\downarrow E \qquad \uparrow E$$

are, respectively, the

lower section   upper section

generated by E, that is the subsets

$$\{x \mid (\exists e \in E)[x \leq e]\} \qquad \{x \mid (\exists e \in E)[x \geq e]\}$$

respectively.

We say that $\uparrow E$ is the *saturation* of E and E is *saturated* if $E = \uparrow E$. □

We write $\downarrow p$ for $\downarrow \{p\}$ and $\uparrow p$ for $\uparrow \{p\}$.

Trivially $\uparrow\uparrow E = \uparrow E$ so the saturation of E is saturated as we would expect.

Notice that for any topological space S and point $p \in S$, we have

$$\downarrow p = p^-$$

by the definition of the specialisation order. However, the same is not generally true for arbitrary subsets of S. Usually

$$\downarrow E \neq E^-$$

although there is a class of topologies (the *Alexandroff topologies*) for which $\downarrow(\cdot)$ and $(\cdot)^-$ agree.

In a topological space the saturation of a subset can be obtained without direct reference to the specialisation order.

Lemma 2.2.3. *Let S be a topological space. For each subset E the intersection*

$$\bigcap\{U \in \mathcal{O}S \mid E \subseteq U\}$$

is the saturation $\uparrow E$ of E.

Proof. If $x \in \uparrow E$ then there exists $e \in E$ such that $e \sqsubseteq x$, or in other words $e^- \subseteq x^-$. Equivalently, we have

$$e \in U \Longrightarrow x \in U$$

for every open set $U \in \mathcal{O}S$. Hence

$$x \in \bigcap\{U \in \mathcal{O}S \mid e \in U\} \subseteq \bigcap\{U \in \mathcal{O}S \mid E \subseteq U\}$$

to give the inclusion $\uparrow E \subseteq \bigcap\{U \in \mathcal{O}S \mid E \subseteq U\}$.

Conversely, if $x \in \bigcap\{U \in \mathcal{O}S \mid E \subseteq U\}$ then

$$E \subseteq U \Longrightarrow x \in U$$

for every open set U. Now there must be some $e \in E$ such that $e \sqsubseteq x$. $\square$

By the above lemma, each open set of a space is saturated. However, the converse need not hold and there are usually plenty of saturated sets which are not open. For instance in a T_1 space every set is saturated.

In an Alexandroff space the converse holds: every saturated set is open. We define these below.

In any poset the family of saturated sets (upper sections) is closed under arbitrary unions and intersections. In particular the saturated sets form a topology. The fact that this collection of sets is closed under all intersections, not just finite ones gives these topologies some special properties.

Definition 2.2.4. Let $(S, \leq)$ be a partial order. The *Alexandroff topology* on S is the topology consisting of all saturated sets. $\square$

Recall the definition of a compact set.

Definition 2.2.5. An *open cover* for a set A is a collection $\mathcal{U}$ of open sets such that

$$A \subseteq \bigcup \mathcal{U}$$

holds.

A *subcover* of an open cover $\mathcal{U}$ for A is a sub-collection $\mathcal{V} \subseteq \mathcal{U}$ which still forms an open cover for A.

A cover $\mathcal{U}$ is *directed* if it is $\subseteq$-directed, that is for each $U, V \in \mathcal{U}$ there is some $W \in \mathcal{U}$ with $U \cup V \subseteq W$.

A subset X of a topological space S is *compact* if every open cover of X has a finite subcover. $\square$

Often it is be convenient to use an equivalent formulation of compactness. We rewrite the definition in terms of directed open covers.

Lemma 2.2.6. *Let S be a topological space. A set $X \subseteq S$ is compact if and only if for every directed open cover $\mathcal{W}$ there exists some $W \in \mathcal{W}$ such that $X \subseteq W$.*

Proof. If X is compact and $\mathcal{W}$ is a directed open cover then there must be a finite cover, $\mathcal{V}$. But by the directedness of $\mathcal{W}$ we must have $\bigcup \mathcal{V} \in \mathcal{W}$.

Conversely, suppose X has the property that for every directed open cover $\mathcal{W}$ there exists some $W \in \mathcal{W}$ such that $X \subseteq W$. If $\mathcal{U}$ is any open cover, then we can transform this into a directed open cover $\mathcal{U}_0$ by adding in all finite unions. Now there exists $U \in \mathcal{U}_0$ such that $X \subseteq U$. However, U is just a finite union of elements of $\mathcal{U}$, giving us a finite subcover of $\mathcal{U}$ as required. $\square$

A major topic of this thesis is an analysis of the compact saturated subsets of a space. We give this family a name.

Definition 2.2.7. For a topological space S, let $\mathcal{Q}S$ be the collection of compact saturated subsets of S. □

The empty set is in $\mathcal{Q}S$. A simple calculation shows that for each point $p \in S$ the saturation $\uparrow p$ is in $\mathcal{Q}S$. This can be generalised.

Lemma 2.2.8. *Let K be a compact subset of a space S. The saturation $\uparrow K$ is in $\mathcal{Q}S$.*

Proof. We need to show that if K is compact, then so is $\uparrow K$. Suppose $\mathcal{U}$ is a directed open cover of $\uparrow K$. Then $\mathcal{U}$ is also a directed open cover of K, so there exists $U \in \mathcal{U}$ with $K \subseteq U$. But then $\uparrow K \subseteq U$ since U is a saturated set including K. Hence $\uparrow K$ is compact. □

It is straightforward to check that the union of two compact sets is compact and the union of two saturated sets is saturated. Thus we have the following.

Lemma 2.2.9. *The union of two compact saturated sets is compact saturated.*

On the other hand, the union of an arbitrary family of compact saturated sets need not be compact saturated. To see this, consider the union of all $\uparrow p$ for every point p of a (non-compact) space S.

Also it is not the case that the intersection of any two compact saturated sets must be compact saturated.

Example 2.2.10. Consider the integers $\mathbb{Z}$ with two extra points l and r below each $m \in \mathbb{Z}$. Set

$$S = \{l, r\} \cup \mathbb{Z}$$

and consider the Alexandroff topology on S. Both the sets

$$\uparrow l = \{l\} \cup \mathbb{Z} \qquad \uparrow r = \{r\} \cup \mathbb{Z}$$

are compact saturated, but

$$\uparrow l \cap \uparrow r = \mathbb{Z}$$

is not, since

$$\{\uparrow m \mid m \in \mathbb{Z}\}$$

is an open covering with no finite subcover. □

In a T_2 space the compact saturated subsets are nicely behaved as the following lemmas show. We do not have this good behaviour in general, and in the next chapter we show one way we might try to fix this situation.

Lemma 2.2.11. *In a T_2 space every compact (saturated) set is closed.*

Proof. In a T_2 space every set is saturated. Let Q be a compact set and p a point not in Q. We show that p lies in an open neighbourhood disjoint from Q.

The T_2 property says that for each point $q \in Q$ there exist open sets U_q, V_q such that

$$p \in U_q \qquad q \in V_q \qquad U_q \cap V_q = \varnothing$$

hold. Then the collection

$$\{V_q \mid q \in Q\}$$

forms an open cover of Q, hence by compactness has a finite subcover

$$\{V_i \mid i \in I\}$$

where I is a finite subset of Q. Now

$$U = \bigcap\{U_i \mid i \in I\}$$

is open, contains p and is disjoint from Q as required. □

This proof also gives us the following result, which should be compared with Definition 2.1.5.

Corollary 2.2.12. *Let S be a T_2 space. For a point p and compact (saturated) set Q not containing p there exist open sets U, V such that*

$$p \in U \qquad Q \subseteq V \qquad U \cap V = \varnothing$$

hold.

Proof. Immediate from the proof of the previous lemma. Set $U = \bigcap\{U_i \mid i \in I\}$ as above and let

$$V = \bigcup\{V_i \mid i \in I\}$$

to give the required properties. □

2.3 The front topology

The front topology of a space S is the smallest topology that makes all the original closed sets clopen. This may look like a very uninteresting topology, but we will see that it has some relevance to the point-free constructions we look at later on.

Definition 2.3.1. The *front space* ${}^{f}S$ of a topological space S has the same points as S but the finer topology $O^{f}S$ generated by

$$\{U \cap X \mid U \in \mathcal{O}S, X \in \mathcal{C}S\}$$

as a base. □

It can be checked that

$$\{U \cap p^- \mid U \in \mathcal{O}S, X \in \mathcal{C}S\}$$

is also a base for the front topology on S. In fact the sets

$$U \cap p^-$$

for $U \in \mathcal{O}S$ form the front open neighbourhoods of $p \in S$.

For $E \subseteq S$ we write

$$E^{\square} \qquad E^{=}$$

for, respectively, the front interior and the front closure of E. These are related by

$$E^{\circ} \subseteq E^{\square} \subseteq E \subseteq E^{=} \subseteq E^{-}$$

and can be distinct.

It is easy to show that

$$p \in E^{=} \iff (\forall U \in \mathcal{O}S)[p \in U \implies E \cap U \cap p^- \neq \varnothing]$$

holds.

At first sight it seems that ${}^{f}S$ is just the discrete space. This is not so. A moment's reflection gives us the following basic facts.

Lemma 2.3.2. *Let S be a topological space.*

If S is T_1 then ${}^{f}S$ is discrete.

If ${}^{f}S$ is discrete then S is T_0.

If S is T_0 then ${}^{ff}S$ is discrete.

In the third clause of Lemma 2.3.2 we used the second front space ${}^{ff}S$. There are examples of sober spaces S such that

$$\mathcal{O}S \qquad \mathcal{O}^{f}S \qquad \mathcal{O}^{ff}S = \mathcal{P}S$$

are distinct.

Example 2.3.3. Let S be the set of real numbers furnished with the topology generated by all sets of the form (p, ∞) for $p \in \mathbb{R}$. Its front topology ${}^{f}S$ is then generated by all sets of the form $(p, q]$ for $p, q \in \mathbb{R}$. This is not discrete. It is, however, T_1 and so by the previous lemma ${}^{ff}S$ is discrete. $\square$

The front construction is a useful technical device. It will reappear in Chapter 6 when we look at the properties of the assembly on a frame. It is also helpful when handling sober spaces.

Theorem 2.3.4. *If S is a sober space, then so is its front space ${}^{f}S$.*

Proof. We see immediately that ${}^{f}S$ is T_0, because S is. It remains to show that every closed irreducible set in ${}^{f}S$ is a point closure.

Suppose that $F \subseteq S$ is closed and irreducible in ${}^{f}S$. Then for every $U, V \in \mathcal{O}S$ and $X, Y \in \mathcal{C}S$

$$\left.\begin{array}{l} F \cap X \cap U \neq \varnothing \\ F \cap Y \cap V \neq \varnothing \end{array}\right\} \implies F \cap X \cap Y \cap U \cap V \neq \varnothing$$

holds by definition. We show first that F^- is closed and irreducible in S.

Suppose that $F^- \cap U \neq \varnothing$ and $F^- \cap V \neq \varnothing$. Then by the properties of the closure operation we have $F \cap U \neq \varnothing$ and $F \cap V \neq \varnothing$ and so $F \cap U \cap V \neq \varnothing$ by irreducibility of F in ${}^fS$. Hence

$$F^- \cap U \cap V \neq \varnothing$$

which is the required property.

We know that S is sober, and so $F^- = \{p\}^-$ for some point $p \in S$. We show next that $p \in F$. For suppose $p \in U \in \mathcal{O}S$. Then $F^- \cap U \neq \varnothing$ and therefore $F \cap U \neq \varnothing$. Hence every front open neighbourhood of p of the form $U \cap p^-$ also meets F. This gives $p \in F^= = F$ as required.

□

Each T_0 space T is embedded as a subspace in its sober reflection ${}^+T$. In particular, if we have $T \subseteq S$ for some sober space S then $T \subseteq {}^+T \subseteq S$. Where can we find this sober space?

Theorem 2.3.5. *Let S be a sober space and let $T \subseteq S$ be an arbitrary subset. Then T is front closed in S precisely when, as a subspace, T is sober.*

Proof. Suppose first that T is front closed. As a subspace the closed subsets of T are those of the form

$$T \cap X$$

for $X \in \mathcal{C}S$. Such a set is a point closure in T precisely when

$$T \cap X = T \cap p^-$$

for some $p \in T$.

Suppose that $T \cap X$ is closed and irreducible in T. Thus

$$\left.\begin{array}{l} T \cap X \cap U \neq \varnothing \\ T \cap X \cap V \neq \varnothing \end{array}\right\} \Longrightarrow T \cap X \cap U \cap V \neq \varnothing$$

for $U, V \in \mathcal{O}S$. Also for such U, V we have

$$(T \cap X)^- \cap U \neq \varnothing \Longrightarrow T \cap X \cap U \neq \varnothing$$

$$(T \cap X)^- \cap V \neq \varnothing \Longrightarrow T \cap X \cap V \neq \varnothing$$

to show that $(T \cap X)^-$ is closed irreducible in S. Since S is sober this gives

$$(T \cap X)^- = p^-$$

for some (unique) $p \in (T \cap X)^-$. We have

$$p \in (T \cap X)^- \subseteq X^- = X$$

and hence $p^- \subseteq X$. Thus

$$T \cap p^- \subseteq T \cap X \subseteq T \cap (T \cap X)^- \subseteq T \cap p^-$$

so that

$$T \cap X = T \cap p^-$$

and it suffices to show that $p \in T$.

For each $U \in \mathcal{O}S$, since $(T \cap X)^- = p^-$, we have

$$\begin{aligned} p \in U &\implies (T \cap X)^- \text{ meets } U \\ &\implies T \cap X \cap U \neq \varnothing \\ &\implies T \cap U \cap p^- \neq \varnothing \end{aligned}$$

and hence

$$p \in T^= = T$$

(since T is front closed), as required.

Conversely, suppose that as a subspace T is sober. We show that $T^= \subseteq T$.

Consider any $p \in T^=$, so that (by the construction of $T^=$)

$$p \in U \iff T \cap U \cap p^- \neq \varnothing$$

for each $U \in \mathcal{O}S$. In particular, with $U = S$ we have

$$T \cap p^- \neq \varnothing$$

and this set is certainly closed in T. We claim that it is irreducible in T. Thus for each $U, V \in \mathcal{O}S$ we have

$$\left.\begin{array}{l} T \cap p^- \cap U \neq \varnothing \\ T \cap p^- \cap V \neq \varnothing \end{array}\right\} \implies p \in U \cap V \implies T \cap p^- \cap U \cap V \neq \varnothing$$

to justify the claim. Since T is sober this gives

$$T \cap p^- = T \cap q^-$$

for some $q \in T \cap p^-$. In particular, $q \in p^-$. But now, for each $U \in \mathcal{O}S$ we have

$$\begin{aligned} p \in U &\implies T \cap p^- \cap U \neq \varnothing \\ &\implies T \cap q^- \cap U \neq \varnothing \\ &\implies q^- \cap U \neq \varnothing \\ &\implies q \in U \end{aligned}$$

so that $p \in q^-$ and hence $p^- = q^-$. Since S is T_0 this gives $p = q \in T$ as required. □

This result gives us a method of locating the sober reflection of a T_0 space.

Corollary 2.3.6. *Let T be a T_0 space and suppose $T \subseteq S$ for some sober space S. Then the sober reflection ${}^+T$ of T is the front closure of T in S.*

Proof. Whatever ${}^+T$ is we must have

$$T \subseteq {}^+T \subseteq S$$

and ${}^+T$ is front closed, by the previous theorem. Thus $T^= \subseteq {}^+T$. Similarly

$$T \subseteq T^= \subseteq S$$

and $T^=$ is sober again by the previous theorem. Thus ${}^+T \subseteq T^=$. □

This useful little result ought to be better known.

Chapter 3

Point-free background

In this chapter we gather together all the basic point-free material. There is almost nothing here that isn't well known.

3.1 Frames

Frames are the main tool for doing point-free topology.

A frame is a partially ordered set with certain other properties. The canonical example of a frame is the collection of open sets on a topological space with the ordering given by set inclusion. However, there are also many examples of frames that are not at all space-like.

We will see that there is a contravariant adjunction between the category of frames and the category of topological spaces.

Definition 3.1.1. A *frame* is a structure

$$\left(A, \leq, \wedge, \bigvee, \top, \bot\right)$$

such that

- $(A, \leq)$ is a complete poset
- $(A, \leq, \wedge, \top)$ is a $\wedge$-semilattice
- $(A, \leq, \bigvee, \bot)$ is a $\bigvee$-semilattice

and the *Frame Distributive Law*

$$a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\}$$

holds for each $a \in A$ and $X \subseteq A$. □

As we will see in Definition 3.1.4, a morphism between frames preserves the distinguished attributes. However, there are other operations that may not be preserved.

Each frame carries a negation and an implication operation.

Definition 3.1.2. Let A be a frame.

The *negation* operation $\neg$ on A is given by

$$\neg a = \bigvee\{x \mid a \wedge x = \bot\}$$

for each $a \in A$.

The *implication* operation $\supset$ on A is given by

$$x \leq (a \supset b) \iff a \wedge x \leq b$$

for each $a, b, x \in A$. □

Of course, we need to show that the implication operation does exist. In fact, we can prove a stronger result.

Theorem 3.1.3. *A complete lattice A is a frame if and only if A carries an implication.*

Proof. ($\Longleftarrow$) Suppose A is a frame. For $a, b \in A$ let $X \subseteq A$ be given by

$$x \in X \iff a \wedge x \leq b$$

(for $x \in A$). Let $c = \bigvee X$. We show that c is $a \supset b$. Trivially for all $x \in A$

$$a \wedge x \leq b \Longrightarrow x \leq c$$

so it suffices to check the converse implication. But the frame distribution law gives

$$a \wedge c = a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\} \leq b$$

and hence

$$x \leq c \Longrightarrow a \wedge x \leq a \wedge c \leq b$$

as required.

($\Longrightarrow$) Now suppose $(A, \leq, \wedge, \bigvee, \top, \bot)$ is a complete lattice with an implication operation. We need to show that the frame distributive law holds. One inequality

$$a \wedge \bigvee X \geq \bigvee \{a \wedge x \mid x \in X\}$$

is trivial. For the other, let

$$b = \bigvee \{a \wedge x \mid x \in X\}$$

so that $a \wedge x \leq b$ for all $x \in X$. Thus $x \leq a \supset b$ for each $x \in X$ and hence

$$\bigvee X \leq a \supset b$$

to give $a \wedge \bigvee X \leq b$ as required. □

Frames are the objects of a category. What are the arrows?

Definition 3.1.4. A *frame morphism*

$$A \xrightarrow{\ f\ } B$$

between frames A and B is a function

$$f\colon A \longrightarrow B$$

which preserves $\leq, \wedge, \bigvee, \top$ and $\bot$. □

This gives us a category Frm of frames and frame morphisms.

For any topological space S its collection of open sets $\mathcal{O}S$ is a frame under the usual operations $\subseteq, \cap, \bigcup$. This is our motivation for the definition. By looking at frames, we can analyse these topologies without referring to the points of the space.

However, there are also many frames which are not topologies (non-spatial frames). In Section 3.3 we will see how to get from a non-spatial frame to a spatial one.

Lemma 3.1.5. *There is a contravariant functor*

$$\mathcal{O} : \mathsf{Top} \longrightarrow \mathsf{Frm}$$

with object assignment and arrow assigment

$$S \longmapsto \mathcal{O}S \qquad\qquad \phi \longmapsto \phi^*$$

respectively for each space S and map ϕ. Here $\mathcal{O}S$ is the topology of S, and ϕ^* is the restriction of the inverse image function $\phi^{\leftarrow}$ to open sets.

Proof. Given

$$\phi\colon S \longrightarrow T$$

in Top, the map

$$\phi^* : \mathcal{O}T \longrightarrow \mathcal{O}S$$

is a frame morphism. It is easy to check that identities and composition are preserved. □

In Section 3.3 we produce a contravariant adjoint to this functor. We also use a different kind of adjunction.

Definition 3.1.6. Let f be a monotone map between two posets A and B. A monotone map g is a *right adjoint* to f if

$$fx \leq y \Longleftrightarrow x \leq gy$$

holds for all $x \in A, y \in B$. □

If a right adjoint exists then it must be unique but not every monotone map has a right adjoint. There is a necessary and sufficient condition for the existence of a right adjoint but we do not need it here. However, every frame morphism does have a right adjoint. We also use the result that for an adjoint pair, the left adjoint preserves arbitrary suprema and the right adjoint preserves arbitrary infima.

We sometimes use the notation

$$f^* \vdash f_*$$

to indicate that f^* is the left adjoint and f_* is the right adjoint.

For every continuous map ϕ in Top from S to T, the corresponding frame morphism

$$\phi^* : \mathcal{O}T \longrightarrow OS$$

must have a right adjoint which we refer to as ϕ_*.

Lemma 3.1.7. *The right adjoint of ϕ^* is given by*

$$\phi_* W = \phi[W']^{-\prime}$$

for every $W \in \mathcal{O}T$

Left and right adjoints combine to produce what we will see are very important gadgets.

Definition 3.1.8. If f^* is an arbitrary frame morphism, the kernel of f^* is given by

$$\ker(f) = f_* \circ f^*$$

where f_* is the right adjoint of f^*. □

Every kernel is a nucleus. These are discussed in Chapter 5.

3.2 Filters

We need a way of adding some extra structure to a frame. Since frames are the tools we use to look at spaces in a point-free way, we need to find an analogue for certain kinds of subsets of the space. In particular, we need something that performs the function of compact saturated sets. This is where filters come in.

Definition 3.2.1. Let A be a frame. A subset F of A is a *filter* if

- $\top \in F$.
- $a \leq b, a \in F \Longrightarrow b \in F$.
- $a, b \in F \Longrightarrow a \wedge b \in F$.

A filter is proper if $\bot \notin F$. □

We are interested in filters with particular properties.

Definition 3.2.2. Let A be a frame. We say that a proper filter F on A is

- *prime* if

$$x \vee y \in F \Longrightarrow x \in F \text{ or } y \in F$$

for every $x, y \in A$.

- *completely prime* if

$$\bigvee X \in F \Longrightarrow X \text{ meets } F$$

for every subset X of A. □

Open filters (or to give them their full title, Scott open filters) are more general gadgets than completely prime filters.

Definition 3.2.3. Let A be a frame. A filter F on A is *(Scott) open* if

$$\bigvee X \in F \Longrightarrow X \cap F \neq \varnothing$$

for every directed set $X \subseteq A$. □

Trivially each completely prime filter is prime and open. We can improve this observation.

Lemma 3.2.4. *A filter is completely prime if and only if it is prime and (Scott) open.*

Proof. The implication

$$\text{completely prime} \Longrightarrow \text{prime} + \text{open}$$

is trivial.

For the other direction, suppose that ∇ is a filter that is prime and open and that $\bigvee X \in \nabla$ for some arbitrary set X. Let $\overline{X}$ be the directed closure of X obtained by adding in all finite joins. Thus

$$\bigvee \overline{X} = \bigvee X \in \nabla$$

and so, because ∇ is open, there exists some $x \in \overline{X}$ with $x \in \nabla$. By the construction of $\overline{X}$ we have $x = \bigvee Y$ for some finite $Y \subseteq X$. Therefore, since ∇ is prime, Y meets ∇ and hence X meets ∇ as required. □

Open filters have other nice properties.

Lemma 3.2.5. *1. The intersection of two open filters is open.*
2. The union of a directed family of open filters is open.

Proof. 1. Suppose F and G are open filters on a frame A and X is a directed subset of A with $\bigvee X \in F \cap G$. Then

$$\bigvee X \in F \qquad \bigvee X \in G$$

so that

$$X \cap F \neq \varnothing \neq X \cap G$$

to give $a, b \in X$ with $a \in F$ and $b \in G$. By the directedness of X there exists $c \in X$ with $a, b \leq c$. So by the upwards closure property $c \in F \cap G$ and hence X meets $F \cap G$ as required.

2. Suppose $\mathcal{F}$ is a directed family of open filters on A and X is a directed subset of A with $\bigvee X \in \bigcup \mathcal{F}$. Then $\bigvee X \in F$ for some $F \in \mathcal{F}$ so X meets F and therefore X meets $\bigcup \mathcal{F}$ as required. □

In the case where $\mathcal{F}$ is a collection of filters that is not directed, taking the union may not give us a filter, so we have to form $\bigvee \mathcal{F}$, the filter generated by the elements of $\bigcup \mathcal{F}$. This operation does not necessarily preserve open filters.

3.3 The point space of a frame

We have seen that if S is a topological space, then its topology $\mathcal{O}S$ is a frame, and that the operation that sends a space to its frame of opens

$$S \longmapsto \mathcal{O}S$$

is a contravariant functor.

However it is not the case that every frame is the topology of some space. The functor $\mathcal{O}$ does not have an inverse, but it does have an adjoint that takes each frame to a topological space. This is the point space construction.

Definition 3.3.1. Let A be a frame. A *character* of A is a frame morphism to the 2-element frame $\mathbf{2}$. □

There is a correspondence between characters of a frame and two other gadgets: prime filters and $\wedge$-irreducible elements.

Definition 3.3.2. Let A be a frame. An element $p \in A$ is $\wedge$-*irreducible* if $p \neq \top$ and

$$x \wedge y \leq p \Longrightarrow x \leq p \text{ or } y \leq p$$

holds for each $x, y \in A$. □

We can use the following to move between three different kinds of widgets.

Lemma 3.3.3. *Let A be a frame. The gadgets*
- *characters of A*
- *completely prime filters of A*
- *$\wedge$-irreducible elements of A*

are in pairwise bijective correspondence.

Proof. It is easy but tedious to check that the widgets constructed in this proof have all the properties claimed.

Suppose p is a character of A. Then

$$\nabla = \{a \in A \mid pa = 1\}$$

is a completely prime filter. Conversely if ∇ is a completely prime filter then the function

$$p\colon A \longrightarrow \mathbf{2}$$

defined by

$$pa = 1 \Longleftrightarrow a \in \nabla$$

is a frame morphism.

Suppose a is a $\wedge$-irreducible element of A. Define $\nabla \subseteq A$ by

$$\nabla = \{x \mid x \nleq a\}$$

to give a completely prime filter. Conversely if ∇ is a completely prime filter then set $a = \bigvee(A - \nabla)$ so that

$$x \in \nabla \Longleftrightarrow x \nleq a$$

and a is irreducible. □

To convert a frame into a space we first produce the points.

Definition 3.3.4. Let A be a frame. The *point space* of A is the collection of all $\wedge$-irreducible elements of A. We write this as $\mathsf{pt}A$. □

By Lemma 3.3.3 we could equally well describe the point space of A as the collection of all completely prime filters of A or the collection of all characters of A. The different ways of describing the points of a frame are useful in different circumstances. Most often we make use of $\wedge$-irreducible elements, but we will use the other descriptions when it suits us.

For $\mathsf{pt}A$ to be a space, it needs a topology.

Definition 3.3.5. Let A be a frame with point space $S = \mathsf{pt}A$ viewed as $\wedge$-irreducible elements. Define

$$U_A(a) = \{p \in S \mid a \not\leq p\}$$

for each element $a \in A$. When the frame in question is clear, we drop the subscript A and just write $U(a)$. □

We could also write this definition in terms of characters or completely prime filters, but this is the one that we will find the most useful. Finite intersections and arbitrary unions of sets of this form are also of this form:

$$U(a) \cap U(b) = U(a \wedge b)$$

and

$$\bigcup\{U(a) \mid a \in X\} = U\Big(\bigvee X\Big)$$

hold for all $a, b \in A$ and $X \subseteq A$. The following lemma is immediate.

Lemma 3.3.6. *Let A be a frame with point space $S = \mathsf{pt}A$. The collection of sets*

$$\{U(a) \mid a \in A\}$$

forms a topology on S and

$$A \xrightarrow{\ U_a(\cdot)\ } \mathcal{O}S$$

is a surjective frame morphsim.

At the beginning of Section 2.2 we defined the specialisation order for a topological space. How does this ordering work on the point space of a frame?

Lemma 3.3.7. *Let A be a frame with point space S. The specialisation order on S is the reverse of the order inherited from A.*

Let's expand on this result. We view the point space S of a frame A as a certain subset of A (the set of $\wedge$-irreducible elements). As such S inherits a comparison $\leq$ from A, but this is *not* the specialisation order. As before we write $\sqsubseteq$ for the specialisation order. Thus

$$p \sqsubseteq q \iff q \leq p$$

for $p, q \in S$.

Each frame A has a surjective frame morphism to the topology $\mathcal{O}S$ of its point space. As such this has a right adjoint.

$$A \underset{U_*}{\overset{U^*}{\rightleftarrows}} \mathcal{O}S$$

Thus $U^* = U_A$ is the notation of Definition 3.3.5. It is useful to have a description of U_*. Each open set of S has the form X' for a closed set X of S. This X is a subset of A and so has an infimum $\bigwedge X$ in A.

Lemma 3.3.8. *In the notation above, for each $X \in \mathcal{C}S$ we have*

$$U_*(X') = \bigwedge X$$

where this infimum is computed in A.

Proof. Using the adjunction $U^* \vdash U_*$, for each $a \in A$ we have

$$\begin{aligned} a \le U_*(X') &\iff U^*(a) \subseteq X' \\ &\iff (\forall p \in S)[a \not\le p \Longrightarrow p \in X'] \\ &\iff (\forall p \in S)[p \in X \Longrightarrow a \le p] \\ &\iff a \le \bigwedge X \end{aligned}$$

which gives the required result. □

Using this we can indicate why we pay special attention to sober spaces.

Lemma 3.3.9. *The point space S of a frame A is sober.*

Proof. The specialisation order of S is the opposite of the comparison inherited from A. In particular, the specialisation order is a partial order and so S is T_0.

Now suppose that $X \in \mathcal{C}S$ is irreducible. Let $p = \bigwedge X$ in A. We show that p is a point of A and that $X = p^-$.

Since $X \neq \varnothing$ there is some $q \in X$, and hence $p \le q < \top$. To show that p is $\wedge$-irreducible we argue by contradiction. Thus suppose

$$a \not\le p \qquad b \not\le p \qquad a \wedge b \le p$$

(for some $a, b \in A$). The first two give $q, r \in X$ with

$$a \not\le q \qquad b \not\le r$$

and hence

$$X \text{ meets } U(a) \qquad X \text{ meets } U(b)$$

at q and r respectively. Since X is irreducible we have

$$X \cap U(a \wedge b) \neq \varnothing$$

and hence some $s \in X$ with $a \wedge b \not\le s$. But now $a \wedge b \le p \le s$ which is the contradiction.

This shows that $p \in S$. To show that

$$X = p^-$$

we first observe that

$$q \in p^- \iff q \sqsubseteq p \iff p \le q$$

for each $q \in S$. Thus an equivalence

$$q \in X \iff p \le q$$

will complete the proof.

The implication ($\Longrightarrow$) is immediate by the definition $p = \bigwedge X$. For the converse we have $X' = U(c)$ for some $c \in A$. But then

$$q \in X \Longleftrightarrow c \leq q$$

to give $c \leq \bigwedge X = p$. Hence

$$p \leq q \Longrightarrow c \leq q \Longrightarrow q \in X$$

as required. □

It can be shown that for a topological space S the point space of $\mathcal{O}S$ is homeomorphic to S exactly when S is sober.

We can use this to give an alternative characterisation of the sober reflection of a space.

Lemma 3.3.10. *Let S be a topological space. The point space of $\mathcal{O}S$ is the sober reflection of S.*

We need not prove this here. The clue is that the closed irreducible subsets of a space S are precisely the complements of the $\wedge$-irreducibles of the topology $\mathcal{O}S$. We find that

$$\begin{array}{ccc} S & \longrightarrow & \mathsf{pt}(\mathcal{O}S) \\ p & \longmapsto & p^{-\prime} \end{array}$$

is the reflection map.

3.4 The Hofmann-Mislove Theorem

Whereas completely prime filters on a topology of a sober space correspond to the points of the space, the usual Hofmann-Mislove Theorem gives us a correspondence between open filters and compact saturated sets.

Here we will generalise this slightly. We start from a frame A and show that an open filter corresponds to a compact saturated set in the point space $S = \mathsf{pt}A$.

For each compact saturated set $Q \in \mathcal{Q}S$ we can obtain an open filter $\nabla(Q)$ on A by

$$x \in \nabla(Q) \Longleftrightarrow Q \subseteq U_A(x)$$

where U_A is the usual reflection morphism from A to $\mathcal{O}S$. We will show that every open filter of A arises in this way from a unique $Q \in \mathcal{Q}S$.

Lemma 3.4.1. *Let F be an arbitrary open filter of A. Let M be the set of maximal members of $A - F$. Then for each $a \in A - F$ there is some $m \in M$ with $a \leq m$.*

Proof. Since F is open the complement $A - F$ is closed under directed suprema, and hence the result follows by a standard application of Zorn's Lemma. □

Next we observe that each $m \in M$ is a point of A. That is, $m \neq \top$ and for $x, y \in A$

$$x \wedge y \leq m \Longrightarrow x \leq m \text{ or } y \leq m$$

holds. This follows by checking the contrapositive.

So, since $M \subseteq S$ we can rephrase Lemma 3.4.1 as follows.

Corollary 3.4.2. *The equivalence*

$$M \subseteq U_A(a) \iff a \in F$$

holds for each $a \in A$.

Proof. Suppose that $a \in F$. Then for each $m \in M$ we have $m \notin F$ so that $a \not\leq m$ and hence $m \in U_A(a)$.

Now suppose that $a \notin F$. Then, by Lemma 3.4.1 there is some $m \in M$ with $a \leq m$, so that $m \notin U_A(a)$ and hence $M \nsubseteq U_A(a)$ as required. □

This result has another more important consequence.

Lemma 3.4.3. *The set* M *is compact (in* S*).*

Proof. Consider any open cover

$$\{U_A(x) \mid x \in X\}$$

of M. In the usual way, we may assume that the index set X is directed. Let $a = \bigvee X$. Then

$$M \subseteq \bigcup\{U_A(x) \mid x \in X\} = U_A(a)$$

and hence $a \in F$. But F is open and X is directed, so that $x \in F$ for some $x \in X$. This gives $M \subseteq U_A(x)$ to produce the required subcover. □

Now let Q be the saturation of M. Since every open set is saturated, we see that Q and M have exactly the same open supersets. In particular, Q is compact, and hence $Q \in \mathcal{Q}S$. Notice also that for each $x \in A$ we have

$$x \in F \iff M \subseteq U_A(x) \iff Q \subseteq U_A(x)$$

so that F arises from Q in the way we want.

Now we show that this is the only compact saturated set attached to F in this way. Suppose there are two, say P, Q. Then

$$P \subseteq U_A(x) \iff Q \subseteq U_A(x)$$

for each $x \in A$. This can be rephrased as

$$(\exists p \in P)[x \leq p] \iff (\exists q \in Q)[x \leq q]$$

for each $x \in A$. Consider any $p \in P$ and let $x = p$. Then we have some $q \in Q$ with $p \leq q$, and hence $q \sqsubseteq p$. Since Q is saturated, this gives $p \in P$ and hence $Q \subseteq P$. Similarly $P \subseteq Q$.

This shows that each open filter F arises from a unique $Q \in \mathcal{Q}S$ in a canonical way.

Lemma 3.4.4. *For* S*,* F *and* Q *all as above,* $Q = S - F$.

Proof. Consider any $q \in Q$. Then, since Q is the saturation of M there is some $m \in M$ such that $m \sqsubseteq q$ in the specialisation order on S. But now in the original ordering on the frame, $q \leq m \notin F$ so that $q \notin F$. This gives $Q \subseteq (S - F)$.

Conversely, suppose $p \in S - F$. Since $p \in A - F$, Lemma 3.4.1 gives some $m \in M$ with $p \leq m$. But then $m \sqsubseteq p$ and hence $p \in Q$ as required. □

We return to these ideas in Chapter 5.

3.5 The frame separation properties regular and fit

In this section we look at two separation properties on frames. One is the point-free version of regularity, the other is a weakening of this condition that has important implications in the analysis of the full assembly.

Definition 3.5.1. A frame A is *regular* if for each $a, b \in A$ with $a \not\leq b$ there exist $x, y \in A$ such that

$$a \vee x = \top \qquad y \not\leq b \qquad x \wedge y = \bot$$

hold.

A frame A is *fit* if for each $a, b \in A$ with $a \not\leq b$ there exist $x, y \in A$ such that

$$a \vee x = \top \qquad y \not\leq b \qquad x \wedge y \leq b$$

hold. □

We will discuss regularity first, then fitness.

It is easy to see that this point-free notion of regularity ties in with the point-sensitive notion of regularity.

Lemma 3.5.2. *A space S is regular if and only if the frame $\mathcal{O}S$ is regular.*

Proof. Suppose the space S is regular, and $M, N \in \mathcal{O}S$ with $M \nsubseteq N$. Then there exists p such that $p \in M, p \notin N$. Now set $X = M'$; then $p \notin X$, so we can apply Definition 2.1.5 to get $U, V \in \mathcal{O}S$ such that

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

hold. But then $M \cup U = S$,

$$p \in V,\ p \notin N \Longrightarrow V \nsubseteq N$$

and $U \cap V = \varnothing$ as required for $\mathcal{O}S$ to be regular.

Conversely, suppose that $\mathcal{O}S$ is regular and $p \notin X \in \mathcal{C}S$. Then we have $X' \nsubseteq p^{-\prime}$ and so there are $U, V \in \mathcal{O}S$ such that

$$X' \cup U = S \qquad V \nsubseteq p^{-\prime} \qquad U \cap V = \varnothing$$

hold. But then

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

as required for S to be regular. □

Closely related to regularity on a frame is the *well-inside* relation.

Definition 3.5.3. Let A be a frame with $a, y \in A$. We say that y is *well-inside* a (and write $y \lessdot a$) if there exists x such that

$$a \vee x = \top \qquad y \leq a \qquad x \wedge y = \bot$$

hold. □

From the definitions, we see that A is regular exactly when for each pair $a \not\leq b$ there is some y such that

$$y \lessdot a \qquad y \not\leq b$$

hold.

Lemma 3.5.4. *A frame A is regular if and only if every element of A is the join of elements well-inside it.*

Proof. Suppose A is regular. For $a \in A$ let

$$b = \bigvee\{y \mid y \lessdot a\}$$

so that $b \leq a$. If $a \not\leq b$ then by the definition of regularity

$$y \lessdot a \qquad y \not\leq b$$

for some $y \in A$ which is a contradiction.

Conversely, suppose

$$a = \bigvee\{y \mid y \lessdot a\}$$

for each $a \in A$. Then

$$a \not\leq b \Longrightarrow (\exists y)[y \lessdot a \text{ and } y \not\leq b]$$

which verifies regularity. □

In Section 8.3 we will stratify this notion of regularity and obtain a corresponding stratified notion of well-inside.

Trivially regularity implies fitness. But fitness is strictly weaker.

Example 3.5.5. The frame of open sets of the 'maximal compact topology' described in Section 10.3 is fit but not regular. □

The fitness property doesn't have an obvious point-sensitive analogue. This is an example of a situation where following the point-free approach gives us tools not available in a point-sensitive setting.

However, fitness does have implications for a space.

It is easy to check that each maximal element of a frame is a point. Suppose $a \in A$ is maximal and $x \wedge y \leq a$ so that $a \vee (x \wedge y) = a$. Now

$$x \not\leq a \Longrightarrow a \vee x > a \Longrightarrow a \vee x = \top$$

and so $\top \wedge (a \vee y) = a$ giving $y \leq a$. This shows that either $x \leq a$ or $y \leq a$ as required.

There may also be points that are non-maximal, and there may be no maximal elements.

Lemma 3.5.6. *Let A be a fit frame. The points (viewed as $\wedge$-irreducible elements) of A are the maximal elements.*

Proof. Suppose that $a \in A$ is $\wedge$-irreducible and $a < b$. Then $b \nleq a$ and because A is fit there exist $x, y \in A$ such that

$$b \vee x = \top \qquad y \nleq a \qquad x \wedge y \leq a$$

hold. The last two conditions imply that $x \leq a$ since a is irreducible. Hence

$$b \vee a \geq b \vee x = \top$$

and so $b = \top$ as required. □

This gives us a method of producing spaces that are both T_1 and sober.

Lemma 3.5.7. *If the frame A is fit then $\mathsf{pt}A$ is T_1 and sober.*

Proof. The open sets of $\mathsf{pt}A$ are

$$U(x) = \{p \mid p \in \mathsf{pt}A,\ x \nleq p\}$$

for each $x \in A$. Now suppose that $p, q \in \mathsf{pt}A$. Then

$$U(q) = \mathsf{pt}A - \{q\} \qquad U(p) = \mathsf{pt}A - \{p\}$$

to give a T_1 separation of p and q. □

In particular every sober space with a fit topology is T_1.

Lemma 3.5.8. *In a space with a fit topology, the three conditions*

$$T_0 \qquad T_1 \qquad \textit{sober}$$

are equivalent.

Proof. We already know that in a space with a fit topology

$$\text{sober} \Longrightarrow T_1 \Longrightarrow T_0$$

so to complete the equivalence, we need to show that if $\mathcal{O}S$ is fit then

$$T_0 \Longrightarrow \text{sober}$$

holds.

Let S be a T_0 space such that $\mathcal{O}S$ is fit. Consider a closed irreducible set $X \subseteq S$, so that X' is $\wedge$-irreducible in $\mathcal{O}S$. Then

$$p \in X \Longrightarrow p^- \subseteq X \Longrightarrow X' \subseteq p^{-\prime}$$

holds for every point $p \in S$. By Lemma 3.5.6, X' is maximal in $\mathcal{O}S$ and so we have $X' = p^{-\prime}$ for each $p \in X$ and hence by the T_0 property $X = p^- = \{p\}$ and X is a point closure. □

Together with Lemma 3.5.7 this gives us the following result.

Corollary 3.5.9. *A T_0 space with a fit topology is T_1 and sober.*

This concludes the background material. We are now ready to move on to the patch constructions themselves.

Chapter 4

The point-sensitive patch construction

The topological spaces most commonly studied are T_2 (Hausdorff). These spaces have a number of desirable properties not necessarily found in non-Hausdorff spaces. One of these is the property that every compact saturated set is closed, as we saw in Lemma 2.2.11. In Section 4.1 we give this property the name 'packed'. This chapter looks at one approach to making an arbitrary space packed.

All the spaces we look at will be T_0. In addition, we will concentrate mainly on sober spaces. However, there are some interesting examples involving non-sober spaces and their sober reflections that we will consider. The details of these appear in Section 10.1.

4.1 Packed spaces

Recall from Lemma 2.2.11 that in a T_2 (Hausdorff) topological space every set is saturated and every compact set is already closed. In more general spaces, this is not usually the case. We need some terminology to refer to spaces where this holds.

Definition 4.1.1. A topological space S is *packed* if every compact saturated set is closed. □

The first question is how the property packed relates to the standard separation properties T_0, T_1 and T_2. It took some time to find an example of a space that was sober and packed but not T_2. In Chapter 11 we examine a class of examples with this property.

Lemma 4.1.2. *A topological space that is T_0 and packed is T_1.*

Proof. The set $(\uparrow p)$ is always compact saturated for every point p, and therefore closed. Now consider two distinct points p, q such that $p \not\sqsubseteq q$ in the specialisation order on S. Then

$$p \in q^{-\prime} \qquad q \in (\uparrow p)'$$

gives a T_1 separation of p and q. □

We can have packed spaces that are not T_0, but the property

$$T_0 + \text{ packed}$$

lies between T_1 and T_2. Notice that a T_0 and packed space need not be sober - the cofinite topology provides the necessary counterexample.

Some of the spaces that we examine later have an even stronger property than packed. It is useful to have a name for this.

Definition 4.1.3. A space S is *tightly packed* if every compact saturated set is finite. □

The examples of Chapter 11 are all tightly packed.

4.2 The point-sensitive patch construction

The point-sensitive patch construction appears in [7] but in a different form.

A space is non-packed if it has at least one compact saturated set which is not closed. In other words, it doesn't have enough closed sets or, equivalently, enough open sets. We attempt to correct the defect by adjoining to the topology new open sets to form a larger topology. We adjoin precisely those open sets that are missing.

The construction needs a little bit of organisation.

Let S be a space with the usual families $\mathcal{O}S, \mathcal{C}S, \mathcal{Q}S$ of respectively: open, closed and compact saturated subsets.

By Lemma 2.2.9 the family

$$\mathsf{pbase} = \{U \cap Q' \mid U \in \mathcal{O}S, Q \in \mathcal{Q}S\}$$

is closed under binary intersections and therefore forms a base for a new topology.

By considering $Q = \varnothing$ we see that pbase includes the original topology, and by letting $U = S$ we see that pbase contains the complement of every compact saturated set.

Definition 4.2.1. For a topological space S, let ${}^{p}S$ be the space with the same points as S and the topology $\mathcal{O}^{p}S$ generated by pbase. □

In other words, $\mathcal{O}^{p}S$ is the smallest topology containing all the original open sets and also the complement of every compact saturated set of S. Note that doing this may create new compact saturated sets which are not closed in ${}^{p}S$.

Using this construction we see that

$$S \text{ packed } \iff {}^{p}S = S$$

holds.

Let's prove some basic results about patch topologies.

Lemma 4.2.2. *Let S be a topological space. Every patch open subset of S is front open.*

Proof. We need to show that for every compact saturated set Q, its complement is front open. Because Q is saturated, we know that Q' is a lower section in the specialisation order, so

$$p \in Q' \Longrightarrow p^{-} \subseteq Q'$$

and hence

$$Q' = \bigcup \{p^{-} \mid p \notin Q\}$$

holds. But for each p, its closure is an original closed set and therefore front open. So Q' is the union of a collection of front open sets and therefore is itself front open. □

The result shows that the patch topology is intermediate between the original topology and front topology. In other words we have

$$\mathcal{O}S \hookrightarrow \mathcal{O}^pS \hookrightarrow \mathcal{O}^fS$$

for every space S.

The next two results relate to separation properties of patch topologies.

Lemma 4.2.3. *The patch space of a T_0 space is T_1.*

Proof. Let S be a T_0 space and consider $p \not\sqsubseteq q \in S$. Then there is some open set $U \in \mathcal{O}S$ such that $p \in U, q \notin U$ giving one half of the T_1 separation. Now observe that $q \notin \uparrow p$ with $\uparrow p \in \mathcal{Q}S$. This gives us a patch open set $(\uparrow p)'$ which contains q but not p. This is the other half of the T_1 separation. □

Lemma 4.2.4. *On a T_1 space the patch operation is idempotent. That is ${}^{pp}S = {}^pS$.*

Proof. Let S be a T_1 space. All subsets of S and ${}^pS$ are saturated, and every compact subset of ${}^pS$ is also compact in S. Thus every compact saturated subset of ${}^pS$ is already patch closed and ${}^{pp}S = {}^pS$ as required. □

Corollary 4.2.5. *For every T_0 space S we have ${}^{ppp}S = {}^{pp}S$.*

Proof. Let S be T_0. By Lemma 4.2.3 the space ${}^pS$ is T_1 and hence applying Lemma 4.2.4 to ${}^pS$ gives the required result. □

This is a nice result. Unfortunately there is a fly in the ointment. In the next section, we see that if we want to keep our spaces sober it is necessary to move to the sober reflection after using the patch operation. Theorem 10.1.10 will show that if we sober up after each patch, then the process can go on forever.

Lemma 4.2.6. *The patch space of a T_2 space is itself.*

Proof. This is just a re-statement of Lemma 2.2.11 which says that in a T_2 space every compact set is closed. □

4.3 The point-sensitive patch is not always sober

Recall that if a space is sober then its front space is also sober. In contrast to this, we will show that the point-sensitive patch space of a sober space is not necessarily sober.

We use a similar method to that used in the analysis of the front topology of a sober space to get some information about the patch space of a sober space. The following results are interesting because they put some restrictions on the nature of the spaces for which the patch space is not sober.

Compare the proof of the following Lemma with the proof of Theorem 2.3.4.

Lemma 4.3.1. *Let S be a sober space and let F be a closed irreducible subset of ${}^pS$ (that is, it is patch closed and patch irreducible). Then F is either a singleton or infinite.*

Proof. The set F is closed and irreducible in ${}^{p}S$ and hence its closure F^- is closed and irreducible in S (because $\mathcal{O}S \subseteq \mathcal{O}{}^{p}S$). The space S is sober, and hence $F^- = p^-$ for a unique $p \in S$. But now, since $p \in F^-$ we have

$$p \in U \in \mathcal{O}S \Longrightarrow F \cap U \cap p^- = F \cap U \neq \varnothing$$

and hence $p \in F^=$. The set F is patch closed, and hence front closed, so that $p \in F^= = F$.

Suppose $F \neq \{p\}$ and, by way of contradiction, suppose F is finite. Thus

$$F = \{p, q_0, \ldots, q_n\}$$

for some points $q_0, \ldots, q_n$ (distinct from p). For each q_i we have $q_i \in F \subseteq p^-$ so that $q_i \sqsubseteq p$ in the specialisation order. Thus

$$p \in F \cap {q_i}^{-\prime}$$

(for otherwise $p \sqsubseteq q_i$ and then $p = q_i$). This shows that each of

$$F \cap (\uparrow p)', F \cap {q_0}^{-\prime}, \ldots, F \cap {q_n}^{-\prime}$$

is non-empty (any q_i is a suitable witness for the first). But

$$F \cap (\uparrow p)' \cap {q_0}^{-\prime} \cap \cdots \cap {q_n}^{-\prime} = \varnothing$$

which contradicts the irreducibility of F in ${}^{p}S$. □

This argument can be refined in different ways to get more information.

Lemma 4.3.2. *The patch space of a T_1 sober space is T_1 and sober.*

Proof. Let S be a T_1 sober space, and let F be a closed irreducible subset of ${}^{p}S$. As in the proof of Lemma 4.3.1, we have $F^- = p^-$ for some $p \in F$. But S is T_1, so that

$$p \in F \subseteq F^- = p^- = \{p\}$$

and hence F is a singleton, as required. □

A second refinement of Lemma 4.3.1 shows that if F is not a singleton then not only is it infinite, it is wide as well.

Lemma 4.3.3. *Let S be a sober space and let F be a closed irreducible subset of ${}^{p}S$ (that is, it is patch closed and patch irreducible). Then F is either a singleton or contains an infinite antichain in the specialisation ordering on S.*

Proof. Let S be a sober space and F a patch closed and irreducible set that is not a singleton.

Observe that the union of a directed family of antichains is an antichain. Hence each subset includes an antichain that is maximal within the subset. Hence it suffices to show that there exists an element $q_0 \in F$ such that any finite antichain containing q_0 is not maximal.

We know from Lemma 4.3.1 that F must be infinite, and that $F^- = p^-$ for some point $p \in F$. Pick an element $r \in F$ with $r \neq p$. If $F \cap (\uparrow r)' = \varnothing$ then r is a minimum for F; in this case choose q_0 with $r < q_0 < p$. Otherwise let $q_0 = r$.

Now the singleton $\{q_0\}$ is an antichain in F with $F \cap (\uparrow q_0)' \neq \varnothing$.

Let $A = \{q_0, \ldots, q_n\}$ be a finite antichain in $F - \{p\}$ with

$$F \cap (\uparrow q_i)' \neq \varnothing$$

for each $0 \leq i \leq n$. (We have already seen the case $n = 0$). We have

$$F \cap {q_i}^{-\prime} \neq \varnothing$$

for each $0 \leq i \leq n$ as witnessed by the point p. Each of ${q_i}^{-\prime}$ and $(\uparrow q_i)'$ is patch open and hence there is some

$$q_{n+1} \in F \cap \bigcap\{{q_i}^{-\prime} \mid i \leq n\} \cap \bigcap\{(\uparrow q_i)' \mid i \leq n\}$$

by the irreducibility of F. Since $q_{n+1} \not\sqsubseteq q_i$ and $q_i \not\sqsubseteq q_{n+1}$ for each $0 \leq i \leq n$, the set $A \cup \{q_{n+1}\}$ is an antichain that extends A.

In this way each finite antichain containing q_0 can be extended by at least one element. But the union of any directed family of antichains is an antichain and so we obtain an infinite antichain in $F - \{p\}$. □

We consider an example where this happens. Here is a sober space which has a non-sober patch space. The details can be found in Section 10.1.

Example 4.3.4. The patch space of the sober reflection of the cocountable topology is not sober. □

In fact, we see that when S is the cocountable topology, the point-sensitive patch of ${}^+S$ is just the cocountable topology on the points of ${}^+S$. Thus by alternately sobering up and taking the point-sensitive patch this construction can be iterated indefinitely.

4.4 Functorial properties of the point-sensitive patch construction

There are some obvious questions regarding the functorality of the point-sensitive patch construction.

Is it possible to view the point-sensitive patch construction

$$S \longmapsto {}^pS$$

as the object assignment of a functor on the category of topological spaces, or on some suitable subcategory? Is it possible to view the continuous map

$${}^pS \longrightarrow S$$

as natural relative to this functor and the identity functor? These two questions can be posed in a more concrete form.

Suppose that

$$T \xrightarrow{\ \phi\ } S$$

is a continuous map between spaces. This gives three sides of a square

$$\begin{array}{ccc} T & \xrightarrow{\ \phi\ } & S \\ \uparrow & & \uparrow \\ {}^{p}T & & {}^{p}S \end{array}$$

where each side is continuous. Under what circumstances is there a continuous map

$${}^{p}T \xrightarrow{\ {}^{p}\phi\ } {}^{p}S$$

which makes the square commute?

As functions both the vertical maps are identity functions. Thus, if there is a map ${}^{p}\phi$, then as a function it is just ϕ. Thus we arrive at the following question.

(?) Under what circumstances is a continuous map

$$T \xrightarrow{\ \phi\ } S$$

also patch continuous, that is continuous relative to the patch topologies?

There appears to be no simple characterisation of patch continuity. Here we set down (without proof) what is known.

First of all we take the simple way out.

Definition 4.4.1. We say a continuous map

$$T \xrightarrow{\ \phi\ } S$$

converts compact saturated sets if $\phi^{\leftarrow}(Q) \in \mathcal{Q}T$ whenever $Q \in \mathcal{Q}S$. □

Thus if ϕ converts compact saturated sets then it is certainly patch continuous. But presumably this sufficient condition for patch continuity is not necessary. Some relevant information can be found in [6].

Let

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

be a frame morphism with its right adjoint. In general f_* preserves arbitrary infima, but need not preserve suprema. We look at those morphisms for which f_* preserves certain suprema.

Definition 4.4.2. For a frame morphism $f^* \vdash f_*$ (as above) the right adjoint f_* is *(Scott)-continuous* if

$$f_*(\bigvee Y) = \bigvee f_*[Y]$$

for each directed subset Y of B. □

Each continuous map

$$T \xrightarrow{\ \phi\ } S$$

gives a frame morphism

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T$$

between the topologies. We may impose the extra condition of (Scott)-continuity on ϕ_*. Of course, this should not be confused with the given continuity of ϕ.

Lemma 4.4.3. *Let ϕ be a continuous map, as above, and suppose the space T is sober. If ϕ_* is (Scott)-continuous, then ϕ converts compact saturated sets, and hence ϕ is patch continuous.*

We need not prove this here. However, a related result will be discussed in Section 7.2.

It is clear that the (Scott)-continuity of ϕ_* is something we should look out for. The following charactersiation is given by Hofmann and Lawson in [6].

Theorem 4.4.4. *Let ϕ be a continuous map, as above, and suppose both the spaces S and T are sober. Then ϕ_* is (Scott)-continuous precisely when both*

- *ϕ converts compact open sets*
- *$Y \in \mathcal{C}T \Longrightarrow \downarrow\phi[Y] \in \mathcal{C}S$*

hold.

Again the details of this proof are not needed here.

Chapter 5

The full assembly

Each frame A has attached to it a second frame

$$A \longrightarrow NA$$

its full assembly. This is an important gadget for understanding and analysing the structure of A. In this and the next chapter we gather together all the relevant results and properties of the assembly. Some results are standard, some results are not new but not as well-known as they should be, and one or two results are new. In this chapter we concentrate on the non-functorial properties.

5.1 Nuclei and related gadgets

The nuclei and related operators of a frame can tell us a lot about the structure of the frame. In this section we gather together all the information we need about these gadgets. However, this is not intended to be a comprehensive account of these matters, so certain well known results will be omitted because they are not needed here.

Definition 5.1.1. Let A be a frame.

(a) An *inflator* on A is a function

$$j : A \longrightarrow A$$

which is inflationary and monotone. That is

$$x \le jx \qquad x \le y \Longrightarrow jx \le jy$$

for all $x, y \in A$.

(b) A *closure operation* on A is an inflator j which is idempotent, that is $j^2 = j \circ j = j$.

(c) A *pre-nucleus* on A is an inflator j such that

$$j(x \wedge y) = jx \wedge jy$$

holds for all $x, y \in A$.

(d) A *nucleus* on A is an pre-nucleus j that is also a closure operation (ie. $j^2 = j$). □

We need some notation for collections of these gadgets. Our main concern is with the family of all nuclei, but the other families have their uses.

Definition 5.1.2. Let A be a frame. Let
(a) IA be the collection of all inflators on A,
(b) PrA be the collection of all pre-nuclei on A,
(c) NA be the collection of all nuclei on A. □

The notation PrA might look a little odd, but we reserve PA for a more important gadget attached to A. An analysis of the gadget PA, which is constructed in Chapter 7 is the central topic of this thesis.

Each family IA, PrA, NA is partially ordered by the pointwise comparison

$$j \leq k \iff (\forall x \in A)[jx \leq kx]$$

(for members j, k of the family). It is easily checked that IA and PrA are closed under composition. NA, however, is not.

The collection of all nuclei is an important attribute of a frame. Let's look at how nuclei naturally arise from frames.

Lemma 5.1.3. *For each frame morphism*

$$A \xrightarrow{\ f\ } B$$

the kernel, $\ker(f)$*, is a nucleus on* A*.*

Proof. We have

$$\ker(f) = f_* \circ f^*$$

where $f^* = f$ and $f^* \dashv f_*$. On general grounds $\ker(f)$ is a closure operation. We are given that f^* is a $\wedge$-morphism. Also, for $x, y \in B$ and $a \in A$ we have

$$\begin{aligned} a \leq f_*(x \wedge y) &\iff f^*a \leq x \wedge y \\ &\iff f^*a \leq x \text{ and } f^*a \leq y \\ &\iff a \leq f_*x \text{ and } a \leq f_*y \\ &\iff a \leq f_*x \wedge f_*y \end{aligned}$$

to show that f_* is a $\wedge$-morphism. Thus the composite $f_* \circ f^*$ is a $\wedge$-morphism to give the required result. □

The kernel of each frame morphism is a nucleus. Conversely, each nucleus is the kernel of an essentially unique surjective morphism. To see this let j be an arbitrary nucleus in the frame A. Let

$$A_j = \{a \in A \mid ja = a\} = j[A]$$

be the set of fixed points of j or, equivalently, the set of values of j. This is a subset of A and so inherits a comparison from A. In fact, A_j is much more than a poset.

The following result is standard (see, for example, Section II.2 of [9]) so we indicate only the important points of the proof. The final part requires the the result that NA is a frame which we will prove later in Section 6.1.

Theorem 5.1.4. *Let j be a nucleus on the frame A.*

(a) The fixed set A_j has finite infima and arbitrary suprema. The infima are the same as in A and suprema are given by

$$\bigvee_j X = j\Big(\bigvee X\Big)$$

for $X \subseteq A_j$.

(b) The fixed set is a frame. The implication on A_j agrees with that on A, that is

$$x \supset_j a = x \supset a$$

for $a, x \in A_j$.

(c) The assignment

$$\begin{array}{ccc} A & \xrightarrow{\ j^*\ } & A_j \\ x & \longmapsto & jx \end{array}$$

is a surjective frame morphsim with $\ker(j^*) = j$.

(d) For each frame morphism

$$f : A \longrightarrow B$$

and each nucleus j on A, if $j \leq \ker(f)$ then there is a unique frame morphism

$$A_j \xrightarrow{\ f_j\ } B$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle j^*}\searrow \quad \nearrow{\scriptstyle f_j} & \\ & A_j & \end{array}$$

commutes.

Proof. (a) This is straightforward.

(b) It is sufficient to show

$$\left.\begin{array}{l} a \in A_j \\ x \in A \end{array}\right\} \Longrightarrow (x \supset a) \in A_j$$

holds for all $a, x \in A$. (In fact it is sufficient to consider only $x \in A_j$.) Let

$$y = x \supset a$$

so that

$$x \wedge y \leq a$$

and hence, assuming $a \in A_j$ we have

$$x \wedge j(y) \leq j(x) \wedge j(y) = j(x \wedge y) \leq j(a) = a$$

to give

$$j(y) \leq x \supset a = y$$

as required.

(c) This assignment j^* is certainly a surjective $\wedge$-morphism. Thus to obtain a frame morphism it will suffice to show

$$j^*\Big(\bigvee X\Big) = \bigvee_j j^*[X]$$

for $X \subseteq A$. For each $x \in X$ we have

$$x \leq \bigvee X$$

so that

$$j(x) \leq j(\bigvee X)$$

to give

$$\bigvee j[X] \leq j(\bigvee X)$$

and hence

$$\bigvee_j j^*[X] = j\Big(\bigvee j[X]\Big) \leq j^2\Big(\bigvee X\Big) = j\Big(\bigvee X\Big) = j^*\Big(\bigvee X\Big)$$

to give one of the required comparisons. The other comparison is immediate.

For $x, y \in A$ we have

$$\begin{aligned} y \leq \ker(j^*) &\iff j^*(y) \leq j^*(x) \\ &\iff y \leq j(x) \end{aligned}$$

to give $\ker(j^*)x = j$.

(d) Since the morphism j^* is surjective, there can be at most one such fill in morphism f_j. In fact, as a function we must have

$$f_j(a) = f(a)$$

for each $a \in A_j$. Thus it suffices to show that this f_j is a morphism. Of the required properties only the comparison

$$f_j\Big(\bigvee_j X\Big) \leq \bigvee f[X]$$

(for $X \subseteq A_j$) is not immediate.

We have

$$j \leq \ker(f) = f_* \circ f^*$$

where $f^* = f$ and $f^* \vdash f_*$. In particular, with $X \subseteq A_j$ and

$$b = \bigvee f[X]$$

we have

$$f^*\Big(\bigvee X\Big) = \bigvee f^*[X] = b$$

so that

$$j\Big(\bigvee X\Big) \le \ker(f)\Big(\bigvee X\Big) = (f_* \circ f^*)\Big(\bigvee X\Big) = f_*(b)$$

to give

$$f_j\Big(\bigvee_j X\Big) = f^*\left(j\Big(\bigvee X\Big)\right) \le b$$

as required. □

Let's look now at the families IA, PrA and NA and their properties. Each is a poset with the pointwise order and each of these posets is complete as shown by the following construction.

Definition 5.1.5. Let A be a frame. For a collection F of inflators, pre-nuclei or nuclei on A, the *pointwise infimum* of F is the function

$$\bigwedge F : A \longrightarrow A$$

given by

$$\Big(\bigwedge F\Big)x = \bigwedge\{fx \mid f \in F\}$$

for each $x \in A$. □

It is straightforward to check that the constructed function $\bigwedge F$ is an inflator on A. Furthermore, it is the infimum of F in the poset IA, so there isn't a conflict of terminology here. Similarly, the pointwise infimum of a collection of pre-nuclei or nuclei is the infimum in PrA or NA respectively.

Lemma 5.1.6. *For a frame A the posets IA, PrA and NA are closed under pointwise infima, and these form the infima in the posets.*

Proof. Much of this is routine. Let's look at the two crucial points.

Consider $F \subseteq PrA$. We show

$$\Big(\bigwedge F\Big)x \wedge \Big(\bigwedge F\Big)y \le \Big(\bigwedge F\Big)(x \wedge y)$$

for $x, y \in A$. But

$$\begin{aligned}
\Big(\bigwedge F\Big)x \wedge \Big(\bigwedge F\Big)y &= \bigwedge\{fx \mid f \in F\} \wedge \bigwedge\{gy \mid g \in F\} \\
&= \bigwedge\{fx \wedge gy \mid f, g \in F\} \\
&\le \bigwedge\{hx \wedge hy \mid h \in F\} \\
&= \bigwedge\{h(x \wedge y) \mid h \in F\} \\
&= \Big(\bigwedge F\Big)(x \wedge y)
\end{aligned}$$

as required. Here the penultimate step uses the fact that each $h \in F$ is a pre nucleus.

Consider $F \subseteq NA$. We show that $\Big(\bigwedge F\Big)$ is idempotent, that is

$$\Big(\bigwedge F\Big)^2 x \le \Big(\bigwedge F\Big)x$$

for each $x \in A$. For

$$\begin{aligned}\left(\bigwedge F\right)^2 x &= \bigwedge\{f\left(\left(\bigwedge F\right)x\right) \mid f \in F\} \\ &\leq \bigwedge\{f(gx) \mid f, g \in F\} \\ &\leq \bigwedge\{h^2x \mid h \in F\} \\ &= \bigwedge\{hx \mid h \in F\} \\ &= \left(\bigwedge F\right)x\end{aligned}$$

as required. Here the penultimate step uses the fact that each $h \in F$ is idempotent. □

Infima are straightforward and can be handled pointwise. Suprema are not so amenable. Remember that a subset $F \subseteq IA$ is directed if it is non-empty and for each $f, g \in F$ there is some $h \in F$ with $f, g \leq h$.

Definition 5.1.7. Let A be a frame. For a directed collection F of inflators on A, the *pointwise supremum* of F is the function

$$\dot{\bigvee} F : A \longrightarrow A$$

given by

$$\left(\dot{\bigvee} F\right) x = \bigvee\{fx \mid f \in F\}$$

for each $x \in A$. □

This construction does not have the same nice properties as the pointwise infimum. First of all, in this case F is required to be a directed set. We could have made the same definition for an arbitrary collection of inflators F, and indeed IA is closed under arbitrary pointwise suprema. However, this is not the case for pre-nuclei.

Lemma 5.1.8. *For any frame A, the posets IA and PrA are closed under directed pointwise suprema, and these form the suprema of directed subsets in the posets.*

In the next construction we pass through an initial section of the ordinals Ord. Later we will see that the ordinals needed are an important measure of the complexity of the situation under consideration.

Definition 5.1.9. Let f be an inflator on the frame A, that is $f \in IA$. The ordinal iterates

$$(f^\alpha \mid \alpha \in \mathsf{Ord})$$

of f are generated by

$$f^0 = \mathrm{id}_A \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda = \bigvee\{f^\alpha \mid \alpha < \lambda\}$$

for every ordinal α and limit ordinal λ. □

Notice that this construction produces an ascending chain of inflators. Since IA is closed under composition each successor step returns an inflator. Also

$$\alpha \leq \beta \Longrightarrow f^\alpha \leq f^\beta$$

so each limit leap returns an inflator. Let's give these observations a bit of status, and add something extra.

Lemma 5.1.10. *Let f be an inflator on the frame A. Then for each ordinal α the iterate f^α is an inflator. Furthermore, if f is a pre-nucleus then each f^α is also a pre-nucleus.*

Proof. Only the last part needs some justification. Furthermore, since pre-nuclei are closed under composition, only the limit case f^λ is not immediate. For that we have

$$\begin{aligned} f^\lambda x \wedge f^\lambda y &= \bigvee\{f^\alpha x \mid \alpha < \lambda\} \wedge \bigvee\{f^\beta y \mid \beta < \lambda\} \\ &= \bigvee\{f^\alpha x \wedge f^\beta y \mid \alpha, \beta < \lambda\} \\ &\leq \bigvee\{f^\gamma x \wedge f^\gamma y \mid \gamma < \lambda\} \\ &= \bigvee\{f^\gamma (x \wedge y) \mid \gamma < \lambda\} \\ &= f^\lambda (x \wedge y) \end{aligned}$$

to give the required result. □

On cardinality grounds, for each inflator f there is at least one ordinal α with $f^{\alpha+1} = f^\alpha$. But then $f^\beta = f^\alpha$ for all $\beta \geq \alpha$. We give the smallest such ordinal a special status.

Definition 5.1.11. For each inflator f on a frame A the *closure ordinal* of f is the least ordinal ∞ such that $f^{\infty+1} = f^\infty$. We call f^∞ the *closure* of f. □

By construction f^∞ is a closure operation. Furthermore, if f is a pre-nucleus then f^∞ is a nucleus. This closure f^∞ has a special relationship to f.

Lemma 5.1.12. *Let f be an inflator on the frame A. Then f^∞ is the smallest closure operation above f. If f is a pre-nucleus then f^∞ is the smallest nucleus above f.*

Proof. Suppose $f \leq g$ for some closure operation g. Then

$$f^\infty \leq g^\infty = g$$

as required.

If f is a pre-nucleus then by Lemma 5.1.10 so is f^∞. A pre-nucleus which is also a closure operation is a nucleus. □

Some examples to illustrate some of the content of this construction can be found in Section 5.4.

This construction allows us to find the suprema in NA of *directed* collections of nuclei. To find suprema of arbitrary collections of nuclei, we need to do a little more work.

Definition 5.1.13. For $J \subseteq NA$ let J° be the compositional closure of J, ie. the set of all

$$j_1 \circ \cdots \circ j_m$$

for $j_1, \ldots, j_m \in J$. □

Notice that J° will not in general be a subset of NA since the set of all nuclei is not closed under composition (for $j_1, j_2 \in NA$, $j_1 \circ j_2$ need not be idempotent). It is, however, a subset of PrA. Notice also that $j_1 \circ j_2 \geq j_1, j_2$ and hence J° is directed, allowing us to take the pointwise supremum.

Lemma 5.1.14. *Let A be a frame. For $J \subseteq NA$,*

$$\bigvee J = \left(\dot{\bigvee} J^\circ\right)^\infty$$

is the supremum of J in NA.

Proof. We have already seen that $\left(\dot{\bigvee} J^\circ\right)^\infty \in NA$. It is also clear that

$$\left(\dot{\bigvee} J^\circ\right)^\infty \geq j$$

for all $j \in J$. It remains to show that this is the *least* upper bound for J. Suppose that for some $g \in NA$ we have $j \leq g$ for all $j \in J$. Then $\left(\dot{\bigvee} J^\circ\right) \leq g$ in PrA and hence $\left(\dot{\bigvee} J^\circ\right)^\infty \leq g$ in NA as required. □

All this shows that the full assembly NA is a complete lattice. In Section 6.1 we will see that it is a frame in its own right, with other interesting properties.

5.2 The u, v and w nuclei

The analysis of the assembly NA of a frame A makes use of some rather simple nuclei. For each $a \in A$ the sections

$$[a, \top] \qquad [\bot, a]$$

are frames and the assignments

$$\begin{array}{ccc} A & \longrightarrow & [a, \top] \\ x & \longmapsto & a \vee x \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & [\bot, a] \\ x & \longmapsto & a \wedge x \end{array}$$

are frame morphisms. The set of

$$A_{\neg\neg} = \{x \in A \mid \neg\neg x = x\}$$

is a complete boolean algebra and the assignment

$$\begin{array}{ccc} A & \longrightarrow & A_{\neg\neg} \\ x & \longmapsto & \neg\neg x \end{array}$$

is a frame morphism. In particular, the composite

$$A \longrightarrow [a, \top] \longrightarrow [a, \top]_{\neg\neg}$$

is a frame morphism.

The kernel of each of these morphisms is a nucleus on A.

These nuclei are crucial to the constructions in this thesis.

Definition 5.2.1. Let A be a frame with $a \in A$. The functions u_a, v_a and w_a on A are given by

$$u_a x = a \vee x \qquad v_a x = (a \supset x) \qquad w_a x = (x \supset a) \supset a$$

for each $x \in A$. □

The functions u_a and v_a are the kernels of the morphisms

$$A \longrightarrow [a, \top] \qquad\qquad A \longrightarrow [\bot, a]$$

respectively. Notice how the kernel of the second morphism uses the implication on A, not the meet as might be expected.

The function w_a is the kernel of the composite morphism

$$A \longrightarrow [a, \top] \longrightarrow [a, \top]_{\neg\neg}$$

which, of course, uses the negation on $[a, \top]$, not that on A. However, for the particular case $a = \bot$ we see that $w_\bot$ is just double negation on A.

These remarks give the following, which can also be proved directly.

Lemma 5.2.2. *For each frame A and $a \in A$, the functions u_a, v_a and w_a are nuclei on A.*

Proof. It is trivial to check the requirements for u_a and v_a; to prove that w_a is a nucleus takes a little more work.

First we need several simple properties of the implication. Recall that by definition

$$z \le (x \supset a) \Longleftrightarrow z \wedge x \le a$$

(for $a, x, z \in A$). As a particular case we have

(1) $a \le (x \supset a)$

and we check that

(2) $x \wedge (x \supset a) = x \wedge a$

holds. Observe that (1) gives

$$x \wedge a \le a \wedge (x \supset a)$$

so it suffices to show the converse comparison. To do this let

$$z = x \wedge (x \supset a)$$

so that $z \le x$ and hence

$$z = z \wedge x \le a$$

thus $z \le x \wedge a$ as required.

From (2) we have

$$x \wedge (x \supset a) \leq a$$

so that the definition of $\supset$ gives

$$x \leq w_a(x)$$

to show that w_a is inflationary.

Next we check

(3) $x \leq y \Longrightarrow (y \supset a) \leq (x \supset a)$

(for $a, x, y \in A$). To see this let $z = (y \supset a)$ so that, assuming $x \leq y$, we have

$$z \wedge x \leq z \wedge y \leq a$$

as required.

Two uses of (3) shows that w_a is monotone.

Since w_a is inflationary we have

$$x \leq w_a(x)$$

and hence

$$w_a(x) \supset a \leq x \supset a$$

by (3). In fact

(4) $w_a(x) \supset a = x \supset a$

holds since

$$x \supset a \leq w_a(x \supset a) = w_a(x) \supset a$$

where the equality follows by unravelling the construction of both compounds.

Two uses of (4) shows that w_a is idempotent, and hence is a closure operation.

To prove now that w_a is a nucleus it suffices to show that

$$w_a(x) \wedge w_a(y) \leq w_a(x \wedge y)$$

for $x, y \in A$. To this end let

$$z = w_a(x) \wedge w_a(y) \wedge (x \wedge y \supset a)$$

so that $z \leq a$ will give the required result. We have

$$z \wedge (x \supset a) \leq a \qquad z \wedge (y \supset a) \leq a \qquad z \wedge x \wedge y \leq a$$

which we use in reverse order. The third gives

$$z \wedge x \leq y \supset a$$

so that

$$z \wedge x \leq z \wedge (y \supset a) \leq a$$

by the second, giving

$$z \leq x \supset a$$

and thus

$$z \leq z \wedge (x \supset a) \leq a$$

by the first. □

These nuclei will be used over and over again in our analysis. Here we take some time to obtain their key properties. Some of the results we will state without proof.

Lemma 5.2.3. *For w_a as defined above*

$$w_a(x \supset a) = (x \supset a)$$

for every $a \in A$.

Proof. We have shown w_a is a nucleus so

$$(x \supset a) \leq w_a(x \supset a)$$

is immediate. To show the converse inequality we observe that

$$\begin{aligned} x \wedge w_a(x \supset a) &\leq w_a(x) \wedge w_a(x \supset a) \\ &= w_a(x \wedge (x \supset a)) \\ &\leq w_a(a) = a \end{aligned}$$

which is sufficient to prove the result. □

Certain compounds are easy to handle.

Lemma 5.2.4. *For every $a, b \in A$ the relations*

$$u_a \vee u_b = u_{a \vee b} \qquad u_a \wedge u_b = u_{a \wedge b}$$

$$v_a \vee v_b = v_{a \wedge b} \qquad v_a \wedge v_b = v_{a \vee b}$$

hold. The suprema and infima on the left of each equality are calculated in NA.

The top left part of this result can be strengthened. For each subset $X \subseteq A$ we have

$$\bigvee \{u_x \mid x \in X\} = u_{\bigvee X}$$

and hence the assignment

$$\begin{array}{ccc} A & \longrightarrow & NA \\ a & \longmapsto & u_a \end{array}$$

is an injective frame morphism.

Lemma 5.2.5. *The nuclei u_a and v_a are complementary in NA; in other words*

$$u_a \vee v_a = \top_{NA} \qquad u_a \wedge v_a = \bot_{NA}$$

for every $a \in A$.

This shows that each element of a frame becomes complemented in its assembly. In the following chapter, Theorem 6.2.2 characterises the assembly as the universal complementation process.

Certain comparisons with u, v or w nuclei are easy to check.

Lemma 5.2.6. *The equivalences*
(a) $u_a \leq j \iff a \leq j\bot$
(b) $v_a \leq j \iff ja = \top$
(c) $j \leq w_a \iff ja = a$
hold for each nucleus j *on* A.

In general

$$j \vee k = (j \circ k)^\infty$$

for $j, k \in NA$ and the closure ordinal can be large. Sometimes we know it is not.

Lemma 5.2.7. *If* $j, k \in NA$ *satisfy*

$$j \circ k \leq k \circ j$$

then

$$k \vee j = k \circ j$$

holds in NA.

Combining with a u or v nucleus is easy.

Corollary 5.2.8. *The relations*
(a) $u_a \vee j = j \circ u_a$
(b) $j \vee v_a = v_a \circ j$
hold for all $a \in A$, $j \in NA$.

In general a nucleus is determined by the elements it fixes. It can also be handled by the intervals it collapses.

Lemma 5.2.9. *Let* A *be a frame. For each pair of elements* $a \leq b$ *the nucleus* $v_a \wedge u_b$ *is the least one which collapses the interval* $[a, b]$.

Finally for this section we show how nuclei can be represented in terms of the basic u, v and w nuclei.

Lemma 5.2.10. *Let* A *be a frame. Each nucleus* j *on* A *can be represented by the following.*
(a) $j = \bigvee\{v_x \wedge u_{jx} \mid x \in A\}$
(b) $j = \bigwedge\{w_{ja} \mid a \in A\}$

Both of these representations are important tools in the analysis of arbitrary nuclei.

5.3 Spatially induced nuclei

For a spatial frame $A = \mathcal{O}S$ there is a class of nuclei which contains all the u and v nuclei described in Section 5.2 and many more as well. These capture the "spatial content" of $\mathcal{O}S$ in a sense to be explained shortly.

Definition 5.3.1. Let S be a topological space. For each $E \in \mathcal{P}S$ we set

$$[E]U = (E \cup U)^\circ$$

for each $U \in \mathcal{O}S$ to obtain a function on $\mathcal{O}S$. □

It is not too hard to see that $[E]$ is a nucleus on the frame $\mathcal{O}S$. For instance, let

$$V = [E]^2 U$$

(for $U \in OS$). Then

$$V = (E \cup [E]U)^\circ \subseteq (E \cup E \cup U)^\circ = (E \cup U)^\circ = [E]U$$

to show that $[E]$ is idempotent. The other required properties are just as easy to check.

Definition 5.3.2. For a topological space S, a nucleus on $\mathcal{O}S$ is *spatially induced* if it has the form $[E]$ for some $E \subseteq S$. □

Shortly we will obtain some of the properties of these special nuclei, but before that we ought to explain the terminology "spatially induced".

Each continuous map

$$T \xrightarrow{\ \phi\ } S$$

between spaces gives a frame morphism

$$\mathcal{O}S \xrightarrow{\ \phi^{\leftarrow}\ } \mathcal{O}T$$

between the topologies. This has a kernel $\ker(\phi^{\leftarrow})$ characterised by

$$V \subseteq \ker(\phi^{\leftarrow})U \iff \phi^{\leftarrow}V \subseteq \phi^{\leftarrow}U$$

for $U, V \in \mathcal{O}S$. What is this nucleus?

Theorem 5.3.3. *Let ϕ be a continuous map, as above. Let $E = S - \phi[T]$, the complement of the range of ϕ. Then $\ker(\phi^{\leftarrow}) = [E]$.*

Proof. For each $U, V \in \mathcal{O}S$ we have

$$\begin{aligned} V \subseteq \ker(\phi^{\leftarrow})U &\iff \phi^{\leftarrow}V \subseteq \phi^{\leftarrow}U \\ &\iff (\forall t \in T)[\phi t \in V \implies \phi t \in U] \\ &\iff (\forall s \in S)[s \in V \cap \phi[T]^{\leftarrow} \implies s \in U] \\ &\iff (\forall s \in S)[s \in V \implies s \in E \cup U] \\ &\iff V \subseteq E \cup U \end{aligned}$$

to give the required result. □

This shows how a "spatially induced" frame morphism gives rise to a spatially induced nucleus. Conversely every spatially induced nucleus arises in this way. To see this consider any space S and subset $E \subseteq S$. Let $T = S - E$ carry the subspace topology, so the insertion

$$T \xrightarrow{\ \phi\ } S$$

is continuous. Then

$$S - \phi[T] = E$$

and hence $[E]$ is the kernel of the embedding.

On a spatial frame, we can determine explicitly the implication operation.

Lemma 5.3.4. *Let S be a topological space. The implication on the spatial frame OS is given by*

$$W \supset M = (W' \cup M)^{\circ}$$

for every $W, M \in \mathcal{O}S$.

Proof. For every $U, W, M \in \mathcal{O}S$ we have

$$U \subseteq (W \supset M) \Longleftrightarrow U \cap W \subseteq M \Longleftrightarrow U \subseteq (W' \cup M)$$

since U is open. This gives the result. $\square$

Each frame carries the u and v nuclei. What are these for a topology?

Lemma 5.3.5. *Let S be a topological space. We have*

$$u_W = [W] \qquad v_W = [W']$$

for each $W \in \mathcal{O}S$.

Proof. (a) For W as above and $M \in \mathcal{O}S$ we see that

$$u_W(M) = W \cup M = (W \cup M)^{\circ} = [W]M$$

as required.

(b) Suppose again that $M \in \mathcal{O}S$. By Lemma 5.3.4 the implication on $\mathcal{O}S$ is given by $W \supset M = (W' \cup M)^{\circ}$. This gives

$$v_W(M) = (W' \cup M)^{\circ} = [W']M$$

as required. $\square$

Each subset E of a space S determines a nucleus E on the topology. However, the nucleus $[E]$ need not determine E.

Recall that as well as the interior E° and closure E^{-} of E we also have the front interior $E^{\square}$ and front closure $E^{=}$.

This next result shows that for each space S there is an insertion

$$\begin{array}{ccc} O^f S & \longrightarrow & NOS \\ E & \longmapsto & [E] \end{array}$$

from the front topology into the assembly of the parent topology. The significance of this will be explained in Chapter 6.

Lemma 5.3.6. *Let S be a topological space. For arbitrary subsets D, E of S, we have $[D] = [E]$ if and only if D and E have the same front interior. In other words*

$$[D] = [E] \Longleftrightarrow D^{\square} = E^{\square}$$

for all $D, E \in \mathcal{P}S$.

Proof. ($\Longleftarrow$) We show that $[D] = [D^\square]$. The inequality $[D^\square] \leq [D]$ is immediate from $D^\square \subseteq D$.

To get the reverse inequality, suppose we have $V \subseteq (D \cup U)$ for open sets $U, V \in \mathcal{O}S$. We will show that $V \subseteq (D^\square \cup U)$.

Since $V \subseteq (D \cup U)$, we see that $V \cap U' \subseteq D$. As $V \cap U'$ is a front open set, this gives $V \cap U' \subseteq D^\square$, and hence $V \subseteq D^\square \cup U$ as required.

This shows that $(D \cup U)^\circ = (D^\square \cup U)^\circ$ and hence that $[D]U = [D^\square]U$ for all open sets U.

($\Longrightarrow$) We just need to show

$$[D] \leq [E] \Longrightarrow D^\square \subseteq E^\square$$

which is enough to prove the result.

Suppose that $[D] \leq [E]$. The sets $U \cap p^-$ (for $U \in \mathcal{O}S$ and $p \in S$) form a base for the front topology. Hence

$$\begin{aligned} p \in D^\square &\Longrightarrow (\exists U \in \mathcal{O}S)[p \in U \cap p^- \subseteq D] \\ &\Longrightarrow (\exists U \in \mathcal{O}S)[p \in U \subseteq D \cup p^{-\prime}] \\ &\Longrightarrow p \in [D]p^{-\prime} \subseteq [E]p^{-\prime} \subseteq E \cup p^{-\prime} \\ &\Longrightarrow p \in E \end{aligned}$$

and so $D^\square \subseteq E$, which gives $D^\square \subseteq E^\square$ since $D^\square$ is front open. $\square$

5.4 The Cantor-Bendixson example

In this section we first set up a well-known point-sensitive construction (the Cantor-Bendixson analysis of a space) and then indicate how it can be generalised to the point-free setting. We state various results, mostly without proof. As we will see, these are intimately connected with the properties of the full assembly NA of a frame A.

This material is not essential for the central topic of the thesis, the study of the patch assembly PA of a frame A. It is included here because later we develop somewhat analogous, and quite new, methods which help with the study of PA. Furthermore, some of the results give a nice contrast between the properties of N and that of P.

Let us recall one way of setting up the Cantor-Bendixson analysis of a space S. To avoid a bit of silliness we assume S is T_0.

For a closed set $X \in \mathcal{C}S$ a point $x \in X$ is *isolated in* X if

$$X \cap U = \{x\}$$

for some open set $U \in \mathcal{O}S$. (If S is not T_0 this definition doesn't do what we want it to.) Let

$$\mathsf{lim}(X)$$

be the set of *limit points* of X, the set of those $x \in X$ which are not isolated in X. We find that $\mathsf{lim}(X)$ is a closed subset of X with

$$\varnothing \subseteq \mathsf{lim}(X) \subseteq X$$

and both extremes can occur. We have

$$X \text{ is discrete } \iff \mathsf{lim}(X) = \varnothing$$

$$X \text{ is perfect } \iff \mathsf{lim}(X) = X$$

where the first is a trivial observation and the second is the definition of 'perfect'.

Trivially lim is a deflationary and monotone operation on $\mathcal{C}S$. In the standard way we can form its ordinal iterates. Thus for each $X \in \mathcal{C}S$ we set

$$\mathsf{lim}^0(X) = X \quad \mathsf{lim}^{\alpha+1}(X) = \mathsf{lim}\left(\mathsf{lim}^{\alpha}(X)\right) \quad \mathsf{lim}^{\lambda}(X) = \bigcap \{\mathsf{lim}^{\alpha}(X) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ.

As usual, for each space S, there is a smallest ordinal ∞ with

$$\mathsf{lim}^{\infty+1} = \mathsf{lim}^{\infty}$$

and this is the *CB-rank* of the space.

For each $X \in \mathcal{C}S$

$$\mathsf{per}(X) = \mathsf{lim}^{\infty}(X)$$

is the *perfect part* of X, that is the largest perfect subset of X. We say X is *scattered* if $\mathsf{per}(X) = \varnothing$.

More often than not these operations are used only for $X = S$. We write

$$S^{(\alpha)} = \mathsf{lim}^{\alpha}(S)$$

as a convenient shorthand.

A property of the operator lim that is not often observed (but is easy to prove) is

$$\mathsf{lim}(X \cup Y) = \mathsf{lim}(X) \cup \mathsf{lim}(Y)$$

for $X, Y \in \mathcal{C}S$. In particular, its dual complement, the operation

$$U \longmapsto \mathsf{lim}(U')'$$

(for $U \in \mathcal{O}S$) is a pre-nucleus on $\mathcal{O}S$. Thus the process of converting lim into per is essentially that of converting a pre-nucleus into its closure, and the CB-rank is the length of that process.

This pre-nucleus can be set up on any frame A. For elements $a, x \in A$ with $a \le x$ we say the interval $[a, x]$ is boolean if, as a frame, it is complemented. That is, if for each $a \le y \le x$ there is some $a \le z \le x$ with $a = y \wedge z$ and $y \vee z = x$.

Definition 5.4.1. Let A be a frame. For each $a \in A$ let $B(a)$ be the set of $x \in A$ where $[a, x]$ is boolean, and let

$$\mathsf{der}(a) = \bigvee B(a)$$

to obtain an operation der on A. □

It turns out that $\mathsf{der}(a) \in B(a)$ that is $[a, \mathsf{der}(a)]$ is the unique largest boolean interval above a.

Lemma 5.4.2. *For each frame A the operation* der *is a pre-nucleus on A.*

This result is not hard to prove, but the details are not important here. The following result shows that der is not a mere curiosity.

Lemma 5.4.3. *For each T_0 space S we have*

$$\mathsf{der}(U) = \mathsf{lim}(U')'$$

for each $U \in \mathcal{O}S$.

For this reason der is called the *CB-derivative* on A, and its closure der^∞ is the *CB-nucleus* on A.

The first part of the next result is immediate from the definition. The proof of the second part is more involved.

Theorem 5.4.4. *We have*

$$A \textit{ is boolean } \iff \mathsf{der}(\bot) = \top$$

$$NA \textit{ is boolean } \iff \mathsf{der}^\infty(\bot) = \top$$

for each frame A.

For a T_0 space S this says that $\mathcal{O}S$ is boolean precisely when S is discrete (which is trivial) and $N\mathcal{O}S$ is boolean precisely when S is scattered (which is not).

5.5 Admissible filters and fitted nuclei

Each nucleus on a frame gives us a filter but this is not a bijective correspondence.

Definition 5.5.1. (1) Let A be a frame. For an element $a \in A$ and nucleus $j \in NA$ we say that j *admits* a if $ja = \top$.

(2) Let $\nabla(j)$ be the set of elements admitted by the nucleus j. Notice that $\nabla(j)$ is a filter on A.

(3) Let A be a frame. A filter on A is *admissible* if it has the form $\nabla(j)$ for some $j \in NA$.

(4) The relation

$$j \sim k \iff \nabla(j) = \nabla(k)$$

is an equivalence relation. We call the equivalence classes *blocks*.

(5) A nucleus is *fitted* if it is the least member of its block. □

There is a one to one correspondence between blocks and fitted nuclei, as shown by the following result.

Lemma 5.5.2. *Let A be a frame. Each block of nuclei has a least member.*

Proof. Let F be an admissible filter on A, and let

$$B = \{j \mid j \in NA, \nabla(j) = F\}$$

so B is the collection of all nuclei that admit exactly the set F. Remembering that infima are computed pointwise in NA, let $k = \bigwedge B$. We claim that $k \in B$, giving us our least element.

Suppose $a \in F$. Then by definition $ja = \top$ for every $j \in B$. This immediately gives $ka = \top$, so that $a \in \nabla(k)$. This shows that $\nabla \subseteq \nabla(k)$. The other inclusion comes from

$$k \leq j \Longrightarrow \nabla(k) \leq \nabla(j) = F$$

where j is any member of B. Thus $\nabla(k) = F$ and hence $k \in B$ as required. □

Some examples of admissible filters are easy to find.

Lemma 5.5.3. *Every principal filter is admissible.*

Proof. Let F be the principal filter $\{x \mid x \geq a\}$ for some $a \in A$. Then the nucleus $j = v_a$ admits F. □

Not every filter is admissible. For instance, suppose A is Boolean. Then each nucleus j has the form u_a for some $a \in A$, or equivalently v_a for some (different) $a \in A$. Then every admissible filter $\nabla(j)$ is principal and when A is infinite there are non-principal filters.

The following result is Lemma 2.4(ii) of [10].

Lemma 5.5.4. *Let A be a frame. Every (Scott) open filter on A is admissible.*

Proof. Let F be an open filter on A and let

$$f = \dot{\bigvee}\{v_a \mid a \in F\}$$

so that for some ordinal ∞ we have $v_F = f^\infty$, and this is the least nucleus that admits F so that $F \subseteq \nabla(f^\infty)$. We wish to show that $\nabla(f^\infty) \subseteq F$. We begin by showing that

$$fx \in F \Longrightarrow x \in F \qquad (\dagger)$$

for each $x \in A$.

The supremum

$$fx = \bigvee\{v_a x \mid a \in F\}$$

is directed and F is an open filter so we have

$$fx \in F \Longrightarrow v_a x \in F$$

for some $a \in F$. Thus $a \in F$ and $a \supset x \in F$ so

$$x \geq a \wedge (a \supset x) \in F$$

and $x \in F$ as required.

Next we prove by ordinal induction that

$$f^\alpha x \in F \Longrightarrow x \in F$$

holds for each ordinal α.

The case $\alpha = 0$ is trivial. The induction step from α to $\alpha + 1$ follows from (†). That just leaves the case for λ a limit ordinal. By definition

$$f^\lambda x = \bigvee\{f^\alpha x \mid \alpha < \lambda\}$$

which is a directed supremum and so

$$f^{\lambda}x \in F \Longrightarrow (\exists \alpha < \lambda)[f^{\alpha}x \in F]$$

because F is open. By the induction hypothesis this implies that $x \in F$.

Thus

$$f^{\infty}x \in F \Longleftrightarrow x \in F$$

for all $x \in A$. In particular

$$f^{\infty}x = \top \Longrightarrow f^{\infty}x \in F \Longrightarrow x \in F$$

and so $\nabla(f^{\infty}) \subseteq F$ as required. □

Not every admissible filter is open, though.

Example 5.5.5. *Consider the filter on $\mathcal{P}\mathbb{N}$ generated by the set of even numbers. This is a principal filter and therefore admissible, but it is not open.*

Not every filter is admissible, but every filter generates a least admissible filter above it.

Definition 5.5.6. Let A be a frame. Suppose F is a filter on A. Then let

$$v_F = \bigvee \{v_a \mid a \in F\}$$

where the supremum is taken in NA. □

Trivially, the nucleus v_F admits each $a \in F$, and so $F \subseteq \nabla(v_F)$. It can be checked that $\nabla(v_F)$ is the least admissible filter above F. Furthermore, v_F is fitted (the least member of its block). In fact, a nucleus is fitted precisely when it is a supremum of v-nuclei.

In some ways these fitted nuclei behave like the v-nuclei. The following should be compared with the lower part of Lemma 5.2.4.

Lemma 5.5.7. *Let A be a frame. The following results hold for all filters F, G and directed families of filters $\mathcal{F}$ on A.*

1. $v_F \wedge v_G = v_{F \cap G}$
2. $v_F \vee v_G = v_{F \vee G}$
3. $\bigvee \{v_F \mid F \in \mathcal{F}\} = v_{\bigcup \mathcal{F}}$

In addition to a least element, some blocks also have a greatest element.

Lemma 5.5.8. *For each $a \in A$ the nucleus w_a is the greatest member of its block.*

Proof. Suppose j is a companion of w_a. It is sufficient to show that $ja = a$ because then $j \leq w_a$ by Lemma 5.2.6.

Let

$$x = ja \qquad y = (x \supset a)$$

so we have $w_a y = y$ by Lemma 5.2.3. Then

$$(y \vee x) \supset a = (y \supset a) \wedge (x \supset a) = (y \supset a) \wedge y = a$$

to give $w_a(y \vee x) = \top$. But then $j(y \vee x) = \top$ since j and w_a are companions. Thus

$$j(y \vee a) = j(y \vee ja) = j(y \vee x) = \top$$

giving $w_a(y \vee a) = \top$. This gives

$$(x \supset a) = y = w_a y = w_a(y \vee a) = \top$$

and $ja = x \leq a$ as required. □

This little known fact will be important in Chapter 9.

A comparison with a fitted nucleus can be made via its filter. The observation

$$j \leq k \Longrightarrow \nabla(j) \subseteq \nabla(k)$$

is trivial. When j is fitted we can strengthen this.

Lemma 5.5.9. *Let A be a frame. Suppose $j \in NA$ is fitted. Then*

$$j \leq k \Longleftrightarrow \nabla(j) \subseteq \nabla(k)$$

holds for all $k \in NA$.

Proof. Suppose $a \in \nabla(j) \subseteq \nabla(k)$ and let x be some element of A. Set $y = v_a x$ so that $a \wedge y \leq x$. Hence

$$y \leq ky = ka \wedge ky = k(a \wedge y) \leq kx$$

which shows that $v_a \leq k$. Thus

$$j = \bigvee\{v_a \mid a \in \nabla(j)\} \leq k$$

as required. □

Recall the definition of a fit frame from Section 3.5. There is a relationship between the separation property fitness and the fitted nuclei.

Theorem 5.5.10. *For each frame A the four conditions*

1. *A is fit*
2. *each nucleus on A is fitted*
3. *each u-nucleus on A is alone*
4. *each u-nucleus on A is minimal in its block*

are equivalent.

Proof. (1) $\Longrightarrow$ (2). Suppose A is fit, and suppose there are unfitted nuclei, so there exist companions j and k with $j \not\leq k$. Then $jc \not\leq kc$ for some $c \in A$. Let $a = jc, b = kc$. Since A is fit, we can find x and y such that

$$a \vee x = \top \qquad y \not\leq b \qquad x \wedge y \leq b$$

hold. Define $z = (y \supset b)$ so that $x \leq z$ and $c \leq b \leq z$ and thus

$$a \leq jc \leq jz \qquad x \leq z \leq jz$$

so $jz \geq a \vee x = \top$ and $kz = \top$ since j and k are companions. But since $y \wedge z \leq b$ we have $ky \leq kb = b$ to give

$$y \leq ky \leq b$$

which contradicts $y \not\leq b$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial.

(4) $\Longrightarrow$ (1). Suppose (4) holds for A, and that $a \not\leq b \in A$. Then $u_a \not\leq w_b$ (since $u_a(\bot) = a$ but $w_b(\bot) = b$). By assumption, u_a is a fitted nucleus, so by Lemma 5.5.9 we know that

$$\nabla(u_a) \nsubseteq \nabla(w_b)$$

holds. Then there exists some $x \in A$ such that

$$a \vee x = \top \qquad w_b x \neq \top$$

hold. Let $y = x \supset b$ so that

$$w_b x = y \supset b \neq \top$$

and $y \not\leq b$. This gives $x \wedge y \leq b$ as required.

□

In other words, fitness is a property that greatly simplifies the structure of the assembly.

5.6 Nuclei associated with open filters

In Section 5.5 we saw that every admissible filter of a frame A is associated with a nucleus

$$v_F = \bigvee\{v_a \mid a \in F\}$$

and that every nucleus is associated with it's admissible filter. In Lemma 5.5.4 we saw that every open filter is admissible.

In this section we take a further look at the nuclei associated with open filters. Recall from Section 3.4 that on a spatial frame there is a correspondence between compact saturated sets and open filters.

In constructing the point-free patch assembly in Chapter 7 we need a point-free gadget to take the place of the compact saturated sets. Open filters are the obvious candidate.

Lemma 5.6.1. *Let A be a frame. Then for all open filters F, G and directed collections $\mathcal{F}$ of open filters we have*

1. $v_F \wedge v_G = v_{F \cap G}$
2. $\bigvee\{v_F \mid F \in \mathcal{F}\} = v_{\bigcup \mathcal{F}}$

and $F \cap G$ and $\bigcup \mathcal{F}$ are open filters.

Proof. This follows immediately from Lemmas 5.5.7 and 3.2.5. □

Each of the fitted nuclei v_F is the supremum over a directed set. In general these are hard to compute. We take the pointwise supremum

$$f_F = \dot{\bigvee}\{v_a \mid a \in F\}$$

(omitting the subscript F where the filter in question is clear) and iterate through the ordinals to give a sequence

$$f^0 = \mathrm{id} \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda = \bigvee\{f^\alpha \mid \alpha < \lambda\}$$

for every ordinal α and limit ordinal λ. This sequence eventually stabilises at f^∞ for some ordinal ∞.

For reasons that will become apparent in Chapter 8 we will concentrate on the sequence obtained by applying each inflator f^α to the bottom element of our frame A. We set

$$d(0) = \bot \qquad d(\alpha+1) = f(d(\alpha)) \qquad d(\lambda) = \bigvee\{f(\alpha) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ.

We can do the same thing in a point-sensitive context. Let S be a topological space. For an open filter F on $\mathcal{O}S$ we have

$$v_F = \bigvee\{v_a \mid a \in F\} = \bigvee\{[U'] \mid Q \subseteq U\}$$

where $Q = \bigcap F$ is the compact saturated set corresponding to F.

This time instead of considering the sequence of opens, it is easier to concentrate on the complementary closed sets. For $Q \in \mathcal{Q}S$ we use the operation $\widehat{Q}$ on $\mathcal{C}S$ given by

$$\widehat{Q}(X) = \bigcap\{(X \cap U)^- \mid Q \subseteq U\}$$

for each $X \in \mathcal{C}S$. Then we set

$$Q(0) = S \qquad Q(\alpha+1) = \widehat{Q}(Q(\alpha)) \qquad Q(\lambda) = \bigcap\{Q(\alpha) \mid \alpha < \lambda\}$$

to give a descending sequence of closed sets.

This construction is similar to the one used in the previous section in the analysis of the Cantor-Bendixson example.

On cardinality grounds, this sequence eventually stabilises at some closed set $Q(\infty)$. We know that

$$Q^- \subseteq Q(\infty)$$

since each closed set $Q(\alpha)$ contains Q. A question we will consider in Chapter 8 is whether we can find conditions that make the difference between Q^- and $Q(\infty)$ either large or small and the consequences this has for the frame and its patch assembly.

5.7 Block structure

It is, perhaps, not a surprise that within the full assembly NA of a frame A there can be some quite complicated blocks. What is a surprise is that this can happen within the assembly of the topology of what seems to be quite a nice space.

To illustrate this let's look at one way of exposing some of the structure of a block.

As usual consider a frame A with $S = \mathsf{pt}(A)$. Let F be an open filter on A and let $Q \in \mathcal{Q}S$ be the corresponding compact saturated set. Thus

$$a \in F \Longleftrightarrow Q \subseteq U(a)$$

for $a \in A$. By Lemma 5.5.4 we know that F is admissible. The corresponding block of NA has a least member v_F. It also has other special members.

Consider the quotient

$$A_F = A_{v_F}$$

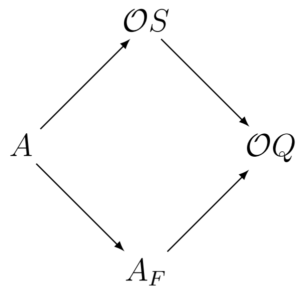
of A by v_F. This has a point space which is easy to locate.

Lemma 5.7.1. *For a frame A and Q, F as above, we have $Q = \mathsf{pt}(A_F)$.*

Proof. The points of A_F are those $p \in S$ with $v_F(p) = p$. If $p \in F$ then $v_F(p) = \top$, and hence $p \notin \mathsf{pt}(A_F)$. Thus $\mathsf{pt}(A_F) \subseteq Q$.

Conversely, consider any $p \in Q$. For each $x \in F$, with $y = (x \supset p)$ we have $y \wedge x \leq p$ and hence $y \leq p$ (since $x \leq p$ requires $p \in F$). Thus $f_F(p) = p$ and hence $v_F(p) = p$, as required. □

This gives us a diagram



which we extend.

Each subset $T \subseteq S$ gives us a quotient

$$A \longrightarrow \mathcal{O}S \longrightarrow \mathcal{O}T$$

by viewing T as a subspace of S. It is routine to check that

$$a \longmapsto \bigwedge\{p \in T \mid a \leq p\}$$

is the kernel of this quotient. In particular, the set $Q \subseteq S$ gives an instance of this. We also use another instance.

By Lemma 3.4.1 the set Q has a set of minimal generators $M \subseteq Q$, namely the set of maximal members of $A - F$. We view M as a subspace of Q to obtain a 3-step quotient

$$A \longrightarrow A_F \longrightarrow \mathcal{O}Q \longrightarrow \mathcal{O}M$$

with kernel given by

$$w_F(a) = \bigwedge\{p \in M \mid a \leq p\}$$

(for $a \in A$). The notation is chosen because of the following result which should be compared with Lemma 5.5.8

Lemma 5.7.2. *The nucleus w_F is the maximum nucleus that admits F.*

Proof. Let j be a nucleus that admits F. Every point $m \in M$ is fixed by j since m is maximal and not in the filter F. We know that

$$ja = \top \Longleftrightarrow a \in F$$

or in other words

$$ja = \top \Longleftrightarrow (\forall m \in M)[j \not\leq m]$$

(see Lemma 3.4.1). Trivially $j(a) = w_F(a) = \top$ for $a \in F$. Now suppose $a \notin F$. Then $a \leq m$ for some $m \in M$ and

$$a \leq m \Longrightarrow ja \leq jm = m$$

so that

$$ja \leq \bigwedge\{p \in M \mid a \leq p\} = w_F(a)$$

which proves the result. □

Every block in NA has a least member, the corresponding fitted nucleus. Such a block need not have a greatest member, or even maximal members. However, for an open filter F the corresponding block

$$[v_F, w_F]$$

is a bounded interval of NA. This gives us a bounded interval

$$I_F = [v_F(\bot), w_F(\bot)]$$

of A. The structure of these blocks are intimately related to the patch properties of A (and other properties).

For each $a \in I_F$ let

$$j_a = v_F \vee u_a$$

to produce a nucleus with $v_F \leq j_a \leq w_F$. A simple calculation gives

$$a \leq b \Longleftrightarrow j_a \leq j_b$$

for $a, b \in I_F$. The implication ($\Longleftarrow$) holds since $I_F \subseteq A_F$, and hence $a = j_a(\bot)$ for each $a \in I_F$. This gives a frame embedding

$$\begin{array}{ccc} I_F & \longrightarrow & [v_F, w_F] \\ a & \longmapsto & j_a \end{array}$$

and hence I_F gives an indication of the complexity of the block.

Of course, it could be that $v_F = w_F$ in which case I_F is a singleton. However, we will produce an example where I_F is quite intricate. In fact, we will produce a spatial example.

Suppose A is spatial, so that $A = \mathcal{O}S$ for some sober space S. For $Q \in \mathcal{Q}S$ we have quotients

$$\mathcal{O}S \longrightarrow (\mathcal{O}S)_F \longrightarrow \mathcal{O}Q \longrightarrow \mathcal{O}M$$

which determine the least, v_F, and greatest, w_F, members of the block, and an intermediate member. In this case we have

$$w_F = [M']$$

the spatially induced nucleus. Similarly $[Q']$ is the intermediate member. Thus we have an interval

$$v_F \leq [Q'] \leq [M'] = w_F$$

of $N\mathcal{O}S$ with a special member $[Q']$. It seems that in general, $[Q']$ can be at either end or somewhere in the middle. The observation that $v_F \leq [Q']$ and that these two nuclei are companions is an important one that we will return to later.

If S is T_1 then $Q = M$ but this does not guarantee the interval is simple.

Example 5.7.3. There is a space S which is T_1, sober and tightly packed. The space has a special point $*$ which controls much of the structure. The set
$\mathbb{S} = S - \{*\}$ is a 'large' tree with many 'large' subtrees. Let F be the filter on $\mathcal{O}S$ given by $Q = \{*\}$, that is the open neighbourhood filter of the point. Then each 'large' subtree of $\mathbb{S}$ produces a member of I_F. □

We will return to this in Section 8.6 where we connect it with a different topic. The example itself is dealt with in Chapter 11 where the precise meaning of 'large' is given.

Chapter 6

Properties of the full assembly

In Chapter 5 we set up most of the basic properties of the assembly of a frame. However, we omitted what are perhaps the two most fundamental properties, namely that the assembly is also a frame and the construction is functorial. This chapter corrects that omission.

6.1 The full assembly is a frame

We know that for a frame A the assembly NA of all nuclei on A is a complete lattice, as is the larger family PrA of pre-nuclei on A. We make use of PrA.

Lemma 6.1.1. *Let A be a frame. For each $f \in PrA, k \in NA$ there is some $l \in PrA$ such that*

$$f \wedge g \leq k \iff g \leq l$$

holds for all $g \in PrA$. Furthermore, l is a nucleus.

Proof. Suppose $f \in PrA$ and $k \in NA$. Let G be the set of all $g \in PrA$ such that $f \wedge g \leq k$. We show that G is closed under composition. For $g, h \in G$ and $x \in A$ we have

$$\begin{aligned}
(f \wedge (g \circ h))x &= fx \wedge g(hx) \\
&\leq f(kx) \wedge g(fx) \wedge g(hx) \text{ since } f(kx), g(fx) \geq fx. \\
&= f(kx) \wedge g(fx \wedge hx) \\
&\leq f(kx) \wedge g(kx) \\
&\leq k^2x = kx
\end{aligned}$$

so that $g \circ h \in G$.

For $f, g \in PrA$ we have $f \circ g \geq f, g$ and hence any subset of PrA that is closed under composition is directed.

Now let $l = \dot{\bigvee} G$, so l is the supremum of G in PrA. As yet we do not know that l is in NA. For each $x \in A$

$$\begin{aligned}
(f \wedge l)x = fx \wedge lx = fx \wedge \bigvee\{gx \mid g \in G\} \\
= \bigvee\{fx \wedge gx \mid g \in G\} \\
\leq kx
\end{aligned}$$

and hence $l \in G$. Thus $l^2 \in G$ since G is closed under composition, which gives $l^2 \leq l$. But l is inflationary, so $l^2 = l$ and $l \in NA$. □

Recall that to show that a complete lattice is a frame it suffices to exhibit an implication operation.

Theorem 6.1.2. *For each frame A, the assembly NA is also a frame.*

Proof. For $f, k \in NA$ we can see that, using the notation of Lemma 6.1.1,

$$(f \supset k) = l$$

with $l \in NA$ and so NA carries an implication and therefore is a frame by Theorem 3.1.3. □

The proof of this result in II.2.5 of [9] uses a more explicit construction of the implication, which provides us with different information. The implication operation is defined as

$$(j \supset k)a = \bigwedge\{j(x) \supset k(x) \mid x \geq a\}$$

(for $a \in A$) and then it is proved that this is in fact an implication on NA.

6.2 N is a functor

The construction N that takes each frame to its assembly together with the canonical embedding

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ a & \longmapsto & u_a \end{array}$$

has some nice properties. In the next main result, Theorem 6.2.2, we show that the morphism n_A universally solves a certain problem, that of complementing elements of A. Before that we make a couple of observations.

By Lemma 5.2.5, for each $a \in A$ the two nuclei u_a and v_a are complementary in NA. Thus the embedding n_A does create complements for elements of A. By Lemma 5.2.10 we have

$$j = \bigvee\{u_{ja} \wedge v_a \mid a \in A\}$$

for each $j \in NA$. This ensures that n_A is epic.

Lemma 6.2.1. *For each frame A and frame morphisms*

$$NA \underset{h}{\overset{g}{\rightrightarrows}} B$$

if $g \circ n_A = h \circ n_A$ then $g = h$. In other words, n_A is epic.

Proof. Consider such a pair g, h with $g \circ n_A = h \circ n_A$. Thus

$$g(u_a) = h(u_a)$$

for each $a \in A$. Also

$$u_a \wedge v_a = \bot_{NA} \qquad u_a \vee v_a = \top_{NA}$$

from which

$$g(v_a) = h(v_a)$$

follows by a simple calculation. Now consider $j \in NA$. We have

$$j = \bigvee \{u_{ja} \wedge v_a \mid a \in A\}$$

so that

$$\begin{aligned} g(j) &= \bigvee \{g(u_{ja}) \wedge g(v_a) \mid a \in A\} \\ &= \bigvee \{h(u_{ja}) \wedge h(v_a) \mid a \in A\} \\ &= h(j) \end{aligned}$$

to give $g = h$ as required. □

This epic property deals with the routine part of the proof of the following.

Theorem 6.2.2. *For each frame A the morphism*

$$A \xrightarrow{\ n_A\ } NA$$

universally solves the complementation problem for A. That is, for each morphism

$$A \xrightarrow{\ f\ } B$$

such that fa has a complement in B for every $a \in A$, there exists a unique morphism $f^\sharp$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle n_A}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & NA & \end{array}$$

commutes.

Proof. Since n_A is epic, there can be at most one such morphism $f^\sharp$. We produce this and its adjoint $f_\flat$.

For each $j \in NA$ set

$$f^\sharp(j) = \bigvee \{f(jx) \wedge f(x)' \mid x \in A\}$$

where $f(x)'$ is the complement of $f(x)$ in B. Trivially the map

$$NA \xrightarrow{\ f^\sharp\ } B$$

is monotone, and an easy calculation shows that it is a $\wedge$-morphism.

For $b \in B$ we have a composite frame morphism

$$A \xrightarrow{\ f\ } B \longrightarrow [b, \top_B]$$

to an interval of B. Let $\langle b \rangle$ be the kernel of this composite, so that

$$y \leq \langle b \rangle x \iff b \vee f(y) \leq b \vee f(x) \iff f(y) \leq b \vee f(x)$$

for all $x, y \in A$. We show that the monotone map

$$\begin{array}{ccc} NA & \xleftarrow{\ f_\flat\ } & B \\ \langle b \rangle & \longleftarrow\!\shortmid & b \end{array}$$

is the right adjoint of $f^\sharp$. In other words we show

$$f^\sharp(j) \leq b \iff j \leq \langle b \rangle$$

for $j \in NA$ and $b \in B$.

Suppose $f^\sharp(j) \leq b$, consider any $x \in A$ and let $y = jx$. We have

$$f(y) \wedge f(x)' \leq f^\sharp(j) \leq b$$

so that

$$j(x) = y \leq f(y) \leq b \vee f(x)$$

to give one implication.

For the converse suppose $j \leq \langle b \rangle$, and consider any $x \in A$. We have

$$jx \leq \langle b \rangle x$$

so that

$$f(jx) \leq b \vee f(x)$$

to give

$$f(jx) \wedge f(x)' \leq b$$

(since $f(x)$ is complemented in B). Letting x range over A gives

$$f^\sharp(j) \leq b$$

as required.

This shows that $f^\sharp$ is a frame morphism.

Finally, for $a, x \in A$ we have

$$f(a \vee x) \wedge f(x)' = (f(a) \vee f(x)) \wedge f(x)' = f(a) \wedge f(x)' \leq f(a)$$

and

$$f(a \vee \top) \wedge f(\top)' = f(a)$$

so that

$$(f^\sharp \circ n_A)(x) = f^\sharp(u_a) = \bigvee \{ f(a \vee x) \wedge f(x)' \mid x \in A \} = f(a)$$

to show that the required triangle does commute. $\square$

Theorem 6.2.3. *The assignment*

$$A \longmapsto NA$$

is the object part of a functor on Frm *and the morphism*

$$A \xrightarrow{n_A} NA$$

is a natural transformation. In other words, for every $A, B \in$ Frm and every frame morphism f, the diagram

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Nf} \\ B & \xrightarrow[n_B]{} & NB \end{array}$$

commutes for some unique morphism Nf.

Proof. In the above diagram, the image of every element of A under $n_B \circ f$ is complemented in NB, and so by Theorem 6.2.2 there is a unique morphism,

$$NA \xrightarrow{Nf} NB$$

which makes the above square commute.

We just need to check that this is a functor; that is, for

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we have $N(g \circ f) = Ng \circ Nf$. But

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Nf} \\ B & \xrightarrow{n_B} & NB \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Ng} \\ C & \xrightarrow{n_C} & NC \end{array}$$

commutes, so $Ng \circ Nf$ must be the unique arrow which makes the outer square commute. Hence $N(g \circ f) = Ng \circ Nf$ as required. □

Theorem 6.2.3 gives us the mere existence of a morphism Nf. A closer look gives us some more concrete information.

Corollary 6.2.4. *For f a frame morphism from A to B*

1. $(Nf)u_a = u_{fa}$
2. $(Nf)v_a = v_{fa}$

for every $a \in A$.

Proof. 1. Follows immediately from the above diagram.

2. We know that u_a and v_a are complementary in NA and that Nf is a frame morphism. Hence

$$u_{fa} \wedge (Nf)v_a = (Nf)(u_a \wedge v_a) = \bot$$

and

$$u_{fa} \vee (Nf)v_a = (Nf)(u_a \vee v_a) = \top$$

so $(Nf)v_a$ is the complement of u_{fa} in NA. Hence $(Nf)v_a = v_{fa}$ as required. □

In time we will obtain more information about the behaviour of N.

6.3 The fundamental triangle of a space

In this section we state and prove the crucial result for finding the point space of the full assembly.

The following lemma is just a special case of Theorem 6.2.2.

Lemma 6.3.1. *For each space S there is a unique frame morphism σ such that*

$$\begin{array}{ccc} \mathcal{O}S & \xrightarrow{\;n_{\mathcal{O}S}\;} & N\mathcal{O}S \\ & {\scriptstyle\iota}\searrow \qquad \swarrow{\scriptstyle\sigma} & \\ & \mathcal{O}^f S & \end{array}$$

commutes.

This is true for any space S, but in practice we almost always use it for the point space of a frame, which is sober.

The morphism σ is a natural transformation as S varies.

We want a concrete description of σ. To find that we first set up a morphism, also called σ, and then show it is the fill in morphism of Lemma 6.3.1.

Definition 6.3.2. For a topological space S let σ be the function from $N\mathcal{O}S$ to $\mathcal{O}^f S$ given by

$$\sigma j = \bigcup \{(jW) - W \mid W \in \mathcal{O}S\}$$

for each $j \in \mathcal{O}S$. □

To show that this σ is a frame morphism we use a more compact description.

Lemma 6.3.3. *For σ as defined in 6.3.2, we have*

$$p \in \sigma j \Longleftrightarrow p \in jp^{-\prime}$$

for every $p \in S$. Furthermore σ is a $\wedge$-morphism.

Proof. For any $p \in S$ if $p \in \sigma j$ then

$$p \in \bigcup\{(jW) - W \mid W \in \mathcal{O}S\}$$

so that

$$p \in (jW) \qquad p \notin W$$

holds for some open set W. This gives

$$W \subseteq p^{-\prime} \qquad p \in jW \subseteq jp^{-\prime}$$

as required.

Conversely suppose that $p \in jp^{-\prime}$. Then setting $W = p^{-\prime}$ we have

$$p \in jW \qquad p \notin W$$

so that $p \in \sigma j$.

For $j, k \in NA$ we have

$$p \in \sigma(j \wedge k) \Longleftrightarrow p \in (j \wedge k)p^{-\prime} \Longleftrightarrow p \in jp^{-\prime} \cap kp^{-\prime} \Longleftrightarrow p \in \sigma j \cap \sigma k$$

to show that σ passes across binary meets. □

Recall that each $E \subseteq S$ gives us a spatially induced nucleus $[E]$ on $\mathcal{O}S$.

Lemma 6.3.4. *For σ as above we have $\sigma([E]) = E^{\square}$ for every $E \in \mathcal{P}S$.*

Proof. Let E be any subset of S. For every open set U we have

$$[E]U - U = (E \cup U)^{\circ} - U \subseteq E^{\square}$$

and so

$$\sigma([E]) = \bigcup\{([E]W) - W \mid W \in \mathcal{O}S\} \subseteq E^{\square}$$

holds.

Now suppose $p \in E^{\square}$. Then there exists $U \in OS$ such that

$$p \in U \cap p^{-\prime} \subseteq E$$

which gives $p \in U \subseteq E \cup p^{-\prime}$ and so $p \in [E]p^{-\prime}$. Therefore

$$p \in [E]p^{-\prime} - p^{-\prime}$$

and $p \in \sigma([E])$. □

By Lemma 6.3.3 the defined

$$N\mathcal{O}S \xrightarrow{\ \sigma\ } \mathcal{O}^f S$$

is at least a $\wedge$-morphism. To show that σ is a frame morphism it suffices to exhibit a right adjoint. We have already found that.

Theorem 6.3.5. *For each space S the pair of assignments*

$$N\mathcal{O}S \underset{[\cdot]}{\overset{\sigma}{\rightleftarrows}} \mathcal{O}^fS$$

form an adjoint pair. Furthermore, σ is a frame morphism.

Proof. We need to check that

$$\sigma j \subseteq E \Longleftrightarrow j \leq [E]$$

holds for each $j \in N\mathcal{O}S$ and $E \in \mathcal{O}^fS$.

Suppose $\sigma j \subseteq E$. Then for every $U \in \mathcal{O}S$ and $p \in S$ we have

$$\begin{aligned} p \in jU &\Longrightarrow p \in (jU) - U \text{ or } p \in U \\ &\Longrightarrow p \in \sigma j \text{ or } p \in U \\ &\Longrightarrow p \in (E \cup U)^\circ \end{aligned}$$

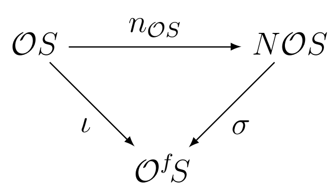
and hence $jU \subseteq [E]U$. Thus $j \leq [E]$ as required.

Conversely, suppose that $j \leq [E]$. Then $\sigma j \subseteq \sigma([E])$. From Lemma 6.3.4 we know that $\sigma([E]) = E^\square$ for all $E \in \mathcal{P}S$. This gives $\sigma j \subseteq E^\square$ and thus $\sigma j \subseteq E$. This shows that $[\cdot]$ is the right adjoint to σ.

On general grounds any monotone function with a right adjoint passes across arbitrary suprema (see Section 3.1). It is trivial to check that σ acts appropriately on $\top_{N\mathcal{O}S}$ and $\bot_{N\mathcal{O}S}$ and is therefore a frame morphism. □

Finally we can prove that this morphism σ we want in Lemma 6.3.1.

Lemma 6.3.6. *The morphism σ defined in 6.3.2 makes the triangle*



commute.

Proof. Suppose that $U \in \mathcal{O}S$. Then

$$(\sigma \circ n_{\mathcal{O}S})U = \sigma([U]) = U^\square = U$$

as required. □

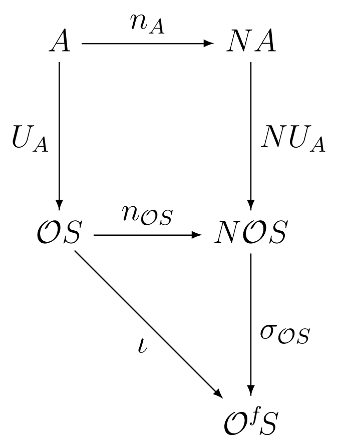
The morphism σ will come in handy later.

6.4 The point space of the full assembly

This section completes the relationship between the full assembly $N\mathcal{O}S$ on a topology and the front topology $\mathcal{O}^f S$.

The following theorem comes from putting the results 6.2.3 and 6.3.1 together.

Theorem 6.4.1. *Let A be a frame and let $S = \mathsf{pt}A$. Then the diagram*



commutes.

We can go further and show that ${}^f S$ is the point space of both NA and $N\mathcal{O}S$. To do this we need to look at a certain class of nuclei - the w nuclei introduced in Definition 5.2.1.

Before that, we need a name for the composite morphism $\sigma_{\mathcal{O}S} \circ NU_A$. We set

$$\Sigma_A = \sigma_{\mathcal{O}S} \circ NU_A$$

and it is easy to show that this is the point space morphism for NA.

The following lemma is extremely important because it identifies the points of the full assembly. We also need to refer to it when we come to look at the point space of the patch assembly.

Lemma 6.4.2. *For each $j \in NA$ the three conditions*

(a) j is $\wedge$-irreducible (in NA)

(b) j is two-valued (every value of j is either $j\bot$ or $\top$)

(c) $a = j\bot$ is $\wedge$-irreducible (in A) and $j = w_a$.

are equivalent.

Proof. We will prove that $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (a)$.

$(a) \Longrightarrow (b)$. Suppose that j is irreducible. We wish to show that j is of the form

$$jx = \begin{cases} \top & \text{if } x \not\leq a \\ a & \text{if } x \leq a \end{cases}$$

where $a = j\bot$. Notice that this is the only possible form a two valued nucleus can take.

We know that

$$u_x \wedge v_x = \bot_{NA} \leq j$$

holds for each $x \in A$. Therefore either $u_x \leq j$ or $v_x \leq j$. Suppose the former. Then

$$x = u_x\bot \leq j\bot = a$$

and hence $jx = ja = a$. This is only possible when $x \leq a$, so suppose $x \nleq a$. Then

$$v_x \leq j \Longrightarrow \top = v_x x \leq jx$$

and we see that

$$x \nleq a \Longrightarrow jx = \top$$

and hence j has the required form.

$(b) \Longrightarrow (c)$. Suppose j is two-valued and $x \wedge y \leq a$ for some $x, y \in A$. Then we have

$$jx \wedge jy \leq j(x \wedge y) \leq a$$

by the form of j. Since $jx, jy \in \{a, \top\}$ then

$$jx = a \quad \text{or} \quad jy = a$$

and hence

$$x \leq a \quad \text{or} \quad y \leq a$$

by the form of j. We know that $a \neq \top$, so this shows that a is $\wedge$-irreducible. Now we just need to show that

$$w_a x = \begin{cases} \top & \text{if } x \nleq a \\ a & \text{if } x \leq a \end{cases}$$

whenever a is $\wedge$-irreducible. Suppose that $x \leq a$. Then

$$(x \supset a) \supset a = \top \supset a = a$$

by the properties of $\supset$. If $x \nleq a$ then

$$(x \supset a) \wedge ((x \supset a) \supset a) \leq a$$

and $x \leq w_a x$ so $(x \supset a) \supset a \nleq a$ Hence, since a is $\wedge$-irreducible $(x \supset a) \leq a$ and then

$$(x \supset a) \supset a = \top$$

as required.

$(c) \Longrightarrow (a)$. We wish to show that w_a is irreducible in NA whenever a is irreducible in A. We have already shown that when a is irreducible then w_a is two-valued. We also have

$$\begin{aligned} k \wedge l \leq w_a &\Longrightarrow ka \wedge la \leq w_a a = a \\ &\Longrightarrow ka \leq a \text{ or } la \leq a \\ &\Longrightarrow k \leq w_a \text{ or } l \leq w_a \end{aligned}$$

by Lemma 5.2.6. Hence w_a is irreducible as required, which completes the proof. □

This gives us an inverse pair of bijections between the point space S of a frame A and the point space T of it's assembly NA. Thus we have

$$S \underset{\psi}{\overset{\phi}{\rightleftarrows}} T$$

where $\phi(p) = w_p$ and $\psi(m) = m\bot$.

So the point space of NA has essentially the same points as S but a different topology. What is this induced topology?

Lemma 6.4.3. *Let A be a frame with point space S and let T be the point space of the assembly NA furnished with the point space topology. Then the induced topology on S that makes*

$$S \underset{\psi}{\overset{\phi}{\rightleftarrows}} T$$

a pair of homeomorphisms is the front topology $\mathcal{O}^f S$.

Proof. The typical open sets of S (with the point space topology) are

$$U_A(x) = \{p \mid p \in S, x \not\leq p\}$$

for $x \in A$. The open sets of T are

$$U_{NA}(j) = \{m \mid m \in T, j \not\leq m\}$$

for each nucleus j.

For each $j \in NA$, we have

$$j = \bigvee \{v_x \wedge u_{jx} \mid x \in A\}$$

and hence the sets

$$U_{NA}(u_x) \qquad U_{NA}(v_x)$$

with $x \in A$ form a subbase of T, since $U(\cdot)$ is a frame morphism.

Now consider that

$$\begin{aligned} \phi^{\leftarrow} U_{NA}(u_x) = \psi\big(U_{NA}(u_x)\big) &= \{\psi m \mid u_x \not\leq m \in T\} \\ &= \{m\bot \mid u_x \not\leq m \in T\} \\ &= \{p \mid x \not\leq p \in S\} \\ &= U_A(x) \end{aligned}$$

using Lemma 5.2.6.

We know that

$$v_x \wedge u_x = \bot \leq m \qquad v_x \vee u_x = \top$$

hold for every $x \in A, m \in T$ and so exactly one of

$$v_x \leq m \qquad u_x \leq m$$

holds. In other words

$$m \in U_{NA}(v_x) \iff m \notin U_{NA}(u_x)$$

for each $m \in T, x \in A$. Hence

$$\phi^{\leftarrow} U_{NA}(v_x) = \psi\big(U_{NA}(v_x)\big) = \psi\big(U_{NA}(u_x)'\big) = U_A(x)'$$

since ψ is a bijective frame morphism.

Similarly, we have $\psi^{\leftarrow} U_A(x) = U_{NA}(u_x)$ and $\psi^{\leftarrow} U_A(x)' = U_{NA}(v_x)$ and hence the sets $U_A(x)$ and $(U_A(x))'$ form a subbase for the induced topology on S, which must therefore be the front topology $\mathcal{O}^f S$. $\square$

Chapter 7

The patch assembly

In this chapter we set up and begin to analyse the central gadget of this thesis, the patch assembly of a frame. We attach to each frame A a certain subframe PA of the full assembly NA. This point-free construction of PA is motivated by the point-sensitive construction of the patch space ${}^{p}S$ of a space S. We will see that $P\mathcal{O}S$ and $\mathcal{O}{}^{p}S$ are related and, in some ways, $P\mathcal{O}S$ is an improvement on ${}^{p}S$.

7.1 The construction

Starting from an arbitrary space S the patch space ${}^{p}S$ is built using

$$\mathcal{O}S \qquad \mathcal{Q}S$$

the families of open subsets and compact saturated subsets of S. We let

$$\mathsf{pbase} = \{U \cap Q' \mid U \in \mathcal{O}S, Q \in \mathcal{Q}S\}$$

to obtain a $\cap$-closed family of subsets of S. We take pbase as the canonical base of a new topology on S. Thus $O{}^{p}S$ is the set of unions of all subfamilies of pbase. These unions, of course, are calculated in $\mathcal{P}S$.

We use an analogous construction to obtain the patch assembly of a frame. Starting from an arbitrary frame A we calculate within the full assembly NA to produce a subframe PA of NA. This is the patch assembly of A. We use the families

$$\{u_a \mid a \in A\} \qquad \{v_F \mid F \text{ an open filter on } A\}$$

of members of NA. The first is an isomorphic copy of A in NA and so is an analogue of $\mathcal{O}S$. By the Hofmann-Mislove results from Section 3.4 the second is a reasonable analogue of $\mathcal{Q}S$.

Definition 7.1.1. For each frame A let

$$\mathsf{PBase} = \{u_a \wedge v_F \mid a \in A,\ F \text{ an open filter on } A\}$$

to obtain a $\wedge$-closed family of members of NA. □

Lemmas 5.2.4 and 5.6.1 show that PBase is $\wedge$-closed. By taking, respectively

$$F = A \qquad a = \top$$

we see PBase contains each u_a for $a \in A$ and each v_F for F an open filter on A.

Definition 7.1.2. For a frame A let PA be the set of suprema of all subfamilies of PBase where these suprema are calculated in NA. This is the *patch assembly* of A. □

By construction PA is closed under arbitrary suprema. A simple exercise (using the frame distributive law for NA) shows that PA is $\wedge$-closed, and hence we have the following.

Theorem 7.1.3. *For each frame A the patch assembly PA is a subframe of NA which includes the canonical image of A.*

This result gives us a small diagram

$$A \longrightarrow PA \hookrightarrow NA$$

where the left hand component is an embedding and the right hand component is an inclusion. Informally we may think of $A \subseteq PA \subseteq NA$ by replacing A by it's canonical image. This construction prompts us to ask several questions.

(Q1) Within the interval

$$A \longrightarrow NA$$

whereabouts can PA occur?

(Q2) Can

$$A \longrightarrow PA$$

be an isomorphism in a non-trivial way?

(Q3) Can $PA = NA$ happen in a non-trivial way?

(Q4) What, if any, is the relationship between this point-free construction and the point-sensitive construction?

(Q5) What is $\mathsf{pt}(PA)$? Is it just ${}^{p}\mathsf{pt}(A)$?

This thesis is an analysis of these and related matters. Some of these questions are answered in full, some only partially.

To conclude this section we give a partial answer to (Q2). The result is taken from [9] but the proof is an improvement.

Theorem 7.1.4. *Suppose A is a regular frame and j is a nucleus on A such that $\nabla(j)$ is open. Then*

$$j = u_d$$

where $d = j\bot$. In other words, regularity implies that the patch assembly is isomorphic to the original frame.

Proof. Since A is regular it is fit, so every block is a singleton and it suffices to show that j and u_d are companions. On general grounds $u_d \leq j$ so we just need to prove

$$jx = \top \Longrightarrow d \vee x = \top$$

for $x \in A$.

Consider any $x \in A$ such that $jx = \top$. Since A is regular, we have

$$x = \bigvee\{y \mid (\exists z)[z \wedge y = \bot \text{ and } z \vee x = \top]\}$$

and this is a directed supremum. Since $\nabla(j)$ is open, j must send one such y to top, and we have

$$jy = \top \qquad z \wedge y = \bot \qquad z \vee x = \top$$

for some $y, z \in A$. This gives

$$d = j\bot = jz \wedge jy = jz$$

and hence

$$d \vee x \geq jz \vee x \geq z \vee x = \top$$

as required. □

Recall that the full assembly embedding

$$A \longrightarrow NA$$

is an isomorphism precisely when A is boolean. Theorem 7.1.4 shows that the patch assembly embedding

$$A \longrightarrow PA$$

is an isomorphism under more interesting circumstances. In Chapter 8 we will give a vast improvement on this result by replacing regularity by a much weaker property.

7.2 Functorality matters

The functorial properties of the point-sensitive patch construction are discussed in Section 4.4. For a continuous map between spaces

$$S \xleftarrow{\ \phi\ } T$$

to be patch-continuous we simply asked that the inverse image function $\phi^{\leftarrow}$ send compact saturated sets $Q \in \mathcal{Q}S$ to compact saturated sets $\phi^{\leftarrow}(Q) \in \mathcal{Q}T$. Each such inverse image function restricts to a frame morphism ϕ^* between the topologies and this has a right adjoint ϕ_*.

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T$$

We observed that the (Scott)-continuity of the right adjoint ϕ_* ensures that ϕ is patch-continuous.

In this section we analyse the functorial properties of the point-free patch construction. We know that each frame morphism

$$A \xrightarrow{\ f\ } B$$

lifts to a morphism Nf between the full assemblies.

$$\begin{array}{ccccc} A & \longrightarrow & PA & \hookrightarrow & NA \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle Nf} \\ B & \longrightarrow & PB & \hookrightarrow & NB \end{array}$$

We wish to obtain a morphism

$$PA \xrightarrow{Pf} PB$$

between the patch assemblies. For this it seems that we need to impose extra conditions on f. Again we take the easy way out and use the 'obvious' condition.

For a filter F on A the direct image $f[F]$ need not be a filter on B but the upward closure

$$fF = \uparrow f[F]$$

is. However, when F is open this image fF need not be.

Example 7.2.1. Consider the real interval $B = [0, 1]$ as a linearly ordered frame. Let $0 < \star < 1$ (say $\star = \frac{1}{2}$) look at

$$A = \{0, \star, 1\}$$

as a subframe and let

$$A \xrightarrow{f} B$$

be the insertion.

The filter $F = \{\star, 1\}$ is completely prime (hence open in A). The image

$$fF = [\star, 1]$$

is not open in B, since $\bigvee [0, \star) = \star$. □

A continuous map need not respect compact saturated sets, and a condition is imposed to achieve functorality. Similarly a frame morphism need not respect open filters, and so we impose an extra condition to achieve this.

Definition 7.2.2. A frame morphism

$$A \xrightarrow{f} B$$

converts open filters if for each open filter F on A the image fF is open on B. □

This condition gives us what we want in a straightforward way. Recall that

$$N(u_a) = u_{fa} \qquad N(v_a) = v_{fa} \qquad N(v_F) = N(v_{fF})$$

for each $a \in A$ and filter F on A. In particular if f converts open filters then

$$F \text{ open in } A \Longrightarrow fF \text{ open in } B$$

(by definition), so that

$$j \in \mathsf{PBase}(A) \Longrightarrow (Nf)j \in \mathsf{PBase}(B)$$

(by a simple calculation), and we have the following.

Lemma 7.2.3. *Suppose the morphism f, as above, converts open filters. Then*

$$j \in PA \Longrightarrow (Nf)j \in PB$$

and hence P acts functorially on this class of arrows.

In other words, when f converts open filters we can define Pf to be the restriction of Nf to PA. This gives a commuting diagram

$$\begin{array}{ccccc} A & \longrightarrow & PA & \hookrightarrow & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Pf} & & \downarrow{\scriptstyle Nf} \\ B & \longrightarrow & PB & \hookrightarrow & NB \end{array}$$

and it is immediate that P passes across compositions as it should.

Our contention is that the point-free patch construction $P(\cdot)$ has some relationship with the point-sensitive patch construction ${}^{p}(\cdot)$, and is perhaps an improvement. A comparison of their actions on arrows illustrates this.

In the next result we consider a frame morphism and its adjoint.

$$A \underset{f_*}{\overset{f = f^*}{\rightleftarrows}} B$$

We also consider a continuous map between spaces

$$S \xleftarrow{\ \phi\ } T$$

and the induced frame morphism

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T$$

with its adjoint.

Theorem 7.2.4. *(a) For a frame morphism f^* as above, if the right adjoint f_* is (Scott)-continuous then f^* converts open filters.*

(b) For a continuous map ϕ, as above, the right adjoint ϕ_* is (Scott)-continuous precisely when ϕ^* converts open filters.

Proof. (a) We have

$$f^*a \le b \Longleftrightarrow a \le f_*b$$

for $a \in A$ and $b \in B$. Consider an open filter F on A and let Y be a directed subset of B with $\bigvee Y \in f^*F$. Thus

$$f^*a \le \bigvee Y$$

for some $a \in F$. But now

$$a \le f_*(\bigvee Y) = \bigvee f_*[Y]$$

by the hypothesis that f_* is (Scott)-continuous. Hence, since $f_*[Y]$ is directed in A we have $a \leq f_* y$ for some $y \in Y$. This gives $f^* a \leq y$ and hence Y meets fF as required.

(b) Suppose first that ϕ_* is continuous. Then, by the previous part we see that ϕ^* converts open filters.

Conversely, suppose ϕ^* converts open filters, consider any directed $\mathcal{W} \subseteq \mathcal{O}T$, and let

$$V = \phi_* \left(\bigcup \mathcal{W} \right)$$

we wish to show

$$V \subseteq \bigcup \phi_*[W]$$

to get Scott-continuity. Consider any $s \in V$. The neighbourhood filter F of s is given by

$$U \in F \iff s \in U$$

(for $U \in \mathcal{O}S$). This filter is open, hence so is ϕF (in $\mathcal{O}T$). But

$$W \in \phi F \iff \phi_* W \in F \iff s \in \phi_* W$$

for each $W \in \mathcal{O}T$. In particular, we have $\bigcup \mathcal{W} \in \phi F$, and hence there is some $W \in \mathcal{W}$ with $W \in \phi F$. Thus

$$s \in \phi_* W \subseteq \bigcup \phi_*[W]$$

as required. □

Consider a continuous map ϕ as above, and suppose the two spaces S and T are sober. We have an associated frame morphism $\phi^* \vdash \phi_*$ between the topologies. Suppose the right adjoint ϕ_* is (Scott)-continuous. Then, by Lemma 4.4.3 the map ϕ converts compact saturated sets, and hence is patch continuous. Also, by Theorem 7.2.4 the morphism ϕ^* converts open filters. Thus we obtain a pair of frame morphisms

$$P\mathcal{O}S \xrightarrow{P(\phi^*)} P\mathcal{O}T \qquad \mathcal{O}^p S \xrightarrow{\phi^*} \mathcal{O}^p T$$

relating the point-free and the point-sensitive constructs. In the next section we will add to this connection.

These observations suggest that perhaps the appropriate frame morphisms to use with the point-free construction $P(\cdot)$ are those which have a continuous right adjoint. This suggestion is wrong.

Theorem 7.2.5. *For each frame A the spatial reflection morphism*

$$A \xrightarrow{U} \mathcal{O}S$$

to the topology of the point space $S = \mathsf{pt}(A)$ converts open filters. It need not have a continuous right adjoint.

Proof. Let F be an open filter on A and let $\nabla = U_A[F]$. We will show that ∇ is open on $\mathcal{O}S$. Consider any directed family $\mathcal{U}$ taken from $\mathcal{O}S$ with $\bigcup \mathcal{U} \in \nabla$. We must show that $\mathcal{U}$ meets ∇.

Consider $X \subseteq A$ given by

$$x \in X \iff U(x) \in \mathcal{U}$$

and so the surjectivity of $U(\cdot)$ gives us

$$\mathcal{U} = \{U(x) \mid x \in X\}$$

and X indexes $\mathcal{U}$ (perhaps with some repetition). We show that X meets F, and hence $\mathcal{U}$ meets ∇.

Let s be the kernel of $U(\cdot)$. Thus

$$y \leq sx \iff U(y) \subseteq U(x)$$

and

$$U(x) = U(sx)$$

for all $x, y \in A$. In particular

$$x \in X \iff sx \in X$$

for $x \in A$. Using this we first check that X is directed. Consider $x, y \in X$. Then $U(x), U(y) \in \mathcal{U}$ and hence (since $\mathcal{U}$ is directed) we have

$$U(x), U(y) \subseteq U(z) = U(sz)$$

for some $z \in X$. Thus $sz \in X$ and the defining property of s gives $x, y \leq sz$, to produce the required upper bound in X.

Now let $a = \bigvee X$. We have

$$U(a) = \bigcup\{U(x) \mid x \in X\} = \bigcup \mathcal{U}$$

so that $U(a) \in \nabla$. Thus

$$U(a) = U(b)$$

for some $b \in F$. But now $sa = sb \in F$ and since $U(sx) = U(x)$ we have

$$sx \in X \iff x \in X$$

and thus

$$\bigvee X = a \in F$$

holds. Since F is open, this gives some $x \in X \cap F$ and hence

$$U(x) \in \mathcal{U} \cap \nabla$$

as required.

To complete the proof we have to produce a frame A where the morphism U_A does not have a continuous right adjoint. This will be given in Example 7.2.7 □

We need a non-trivial frame with no points. There are some complicated frames of this kind, but here is a simple way of producing one.

Lemma 7.2.6. *Let S be any T_1 sober space with no isolated points, and let $(\mathcal{O}S)_{\neg\neg}$ be the boolean algebra of regular open sets of S (the quotient of the frame $\mathcal{O}S$ under the nucelus given by double negation). The frame $(\mathcal{O}S)_{\neg\neg}$ has no points.*

Proof. We view points of $(\mathcal{O}S)_{\neg\neg}$ as characters

$$(\mathcal{O}S)_{\neg\neg} \xrightarrow{\ p\ } 2$$

and suppose that there exists such a point, p. Then we can pull it back to a point

$$\mathcal{O}S \xrightarrow{\ \neg\neg\ } (\mathcal{O}S)_{\neg\neg} \xrightarrow{\ p\ } 2$$

of $\mathcal{O}S$. Hence for some $p \in S$ we have

$$p^{-\prime} \in \mathsf{pt}(\mathcal{O}S)$$

which goes to a point of $(\mathcal{O}S)_{\neg\neg}$. Hence $\neg\neg(p^{-\prime}) = \{p\}^{\circ-\prime}$ is a point of $(\mathcal{O}S)_{\neg\neg}$. But $\{p\}^{\circ} = \varnothing$ since S has no isolated points, and so $\neg\neg(p^{-\prime}) = S$ which is not irreducible in $(\mathcal{O}S)_{\neg\neg}$. ☐

For the next example, we need a non-trivial frame with no points. To get this, we just apply the above Lemma to the space $S = \mathbb{R}$ with the metric topology.

Example 7.2.7. Let B be a non-trivial frame with no points. Add a copy of $\mathbb{N}$ below B, so that we have a tail X consisting of ω elements. This forms a frame, A. We claim that for this frame U_A does not have a continuous right adjoint.

To show this, let s be the kernel of the nucleus U_A. We call s the *spatial nucleus* of A. Notice that

$$U(x) \subseteq U(a) \iff x \leq sa$$

for $x, a \in A$. The spatial nucleus takes each element of A to the infimum (in A) of the points above it.

By way of contradiction suppose U has a continuous right adjoint. Then its kernel s is continuous (since s is the composite of U and its right adjoint). In other words

$$s\left(\bigvee X\right) = \bigvee s[X]$$

for each directed subset X of A.

Consider the tail X of A. Then $\bigvee X = b$, the bottom element of B. Thus

$$s\left(\bigvee X\right) = sb = \top$$

since there are no points (of A) above b. But each element of X is a point of A, hence

$$\bigvee s[X] = \bigvee X = b$$

which is the contradiction. ☐

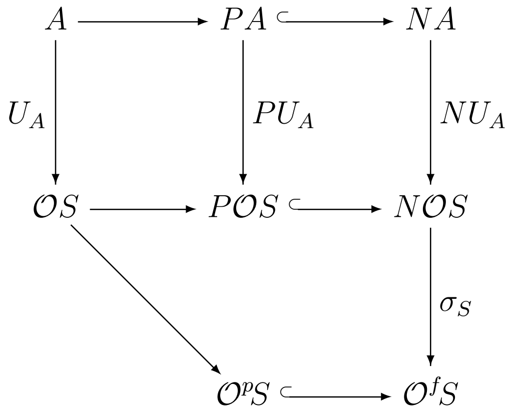
There is a positive side to Theorem 7.2.5 – it enables us to hit U_A with the functor P and so obtain a morphism

$$PA \xrightarrow{\ P(U_A)\ } P\mathcal{O}S$$

between the associated patch assemblies. We use this in the next section.

7.3 The full patch assembly diagram

We can put all the connections so far together into one large diagram.



The top rectangle of this diagram is just the commuting diagram following Lemma 7.2.3. The lower section

$$\mathcal{O}S \hookrightarrow \mathcal{O}^pS \hookrightarrow \mathcal{O}^fS$$

has been dealt with in Section 4.2 and is a consequence of Lemma 4.2.2.

The morphism

$$N\mathcal{O}S \xrightarrow{\ \sigma\ } \mathcal{O}^fS$$

is the point space morphism from the full assembly to the topology of its point space, the front topology. This is dealt with in Section 6.4.

The obvious question is whether there is a morphism that fits between $P\mathcal{O}S$ and $\mathcal{O}^pS$ that makes the resulting cells commute.

Restricting σ to $P\mathcal{O}S$ will do the trick.

Lemma 7.3.1. *Let S be a space with topology $\mathcal{O}S$. For each open filter F on $\mathcal{O}S$ we have Q with $F = \nabla(Q)$,*

$$\sigma(v_F) = Q'$$

where Q is the corresponding compact saturated set, that is $F = \nabla(Q)$.

Proof. We have seen in Section 5.7 that v_F and $[Q']$ are companions and admit the same elements.

Since v_F is the minimum element in its block we have $v_F \leq [Q']$ and thus

$$\sigma(v_F) \subseteq \sigma([Q']) = Q'$$

since Q' is front open.

To show that $Q' \subseteq \sigma(v_F)$, suppose that $p \in Q'$. Then $p^- \subseteq Q'$ by saturation of Q, so $Q \subseteq p^{-\prime}$ and thus $p^{-\prime} \in F$. Therefore

$$p \in v_F(p^{-\prime}) = \top$$

and so $p \in \sigma(v_F)$ by Lemma 6.3.3 giving

$$Q' \subseteq \sigma(v_F)$$

as required. $\square$

Lemma 7.3.2. *Let S be a space with topology $\mathcal{O}S$. The restriction of the frame morphism σ to $P\mathcal{O}S$ gives a frame morphism π from PA to $\mathcal{O}^pS$. Furthermore, the morphism π is surjective.*

Proof. The frame $P\mathcal{O}S$ is generated by nuclei of the form

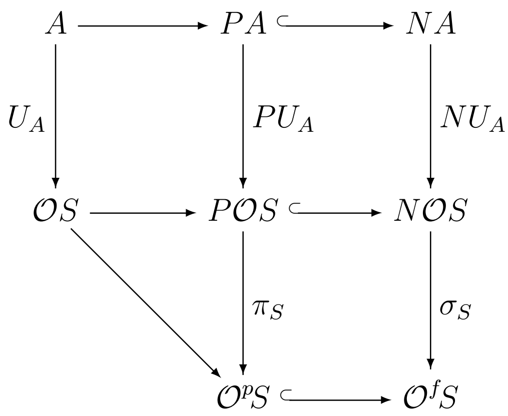
$$[U] \wedge v_F$$

where $U \in \mathcal{O}S$ and F is an open filter on $\mathcal{O}S$. We have $\sigma([U]) = U$, since U is open, and $F = \nabla(Q)$ for some $Q \in \mathcal{Q}S$ so that $\sigma(v_F) = Q'$ by Lemma 7.3.1. Thus

$$\sigma\left([U] \wedge v_F\right) = U \cap Q'$$

and these sets form a base for the patch topology $\mathcal{O}^pS$. □

This gives us the full patch diagram.



The frame morphism π need not be an isomorphism although it seems to be in most natural examples. However, π is not injective for some of the examples in Chapter 11. In fact if S is packed then ${}^pS = S$ so

$$\mathcal{O}S \rightleftarrows P\mathcal{O}S$$

composes to give the identity at $\mathcal{O}S$, but π still need not be an isomorphism.

Chapter 8

A hierarchy of separation properties

In this chapter we obtain the vast improvement of Theorem 7.1.4 promised in Section 7.1. A detailed analysis of the proof of that result leads to an ordinal indexed hierarchy of separation properties lying between T_1 and T_2.

8.1 Patch triviality

The point-sensitive patch construction acts on a space to make all the compact saturated sets closed. On a packed space, where all compact saturated sets are already closed, it does nothing. How does the point-free patch construction behave when applied to the topology of a packed space? Does it, as we might initially expect, give us a frame isomorphic to the original, or can it behave differently?

In Section 7 we set up the canonical embedding

$$\begin{array}{ccc} A & \longrightarrow & PA \\ a & \longmapsto & u_a \end{array}$$

for an arbitrary frame. We ask when this embedding is an isomorphism. For the time being we make the following definition.

Definition 8.1.1. A frame A is *patch trivial* if the embedding

$$A \longrightarrow PA$$

is an isomorphism. □

Later we will find a way of characterising patch triviality, and this leads to some better terminology, but for the moment it is useful to refer to it in this way.

The corresponding question for the full assembly is not very interesting. We know that

$$A \longrightarrow NA$$

is an isomorphism exactly when the frame A is boolean. By contrast the question for the patch assembly is going to produce some very interesting frames.

Question 8.1.2. *Under what circumstances is a frame A patch trivial?*

Recall that the patch assembly PA of a frame A is generated by the trivial (sometimes "closed") nuclei u_a (for $a \in A$) and the fitted nuclei v_F associated with the open filters F of A. Thus for a frame A to be patch trivial, for every open filter F there must be some $d \in A$ such that

$$v_F = u_d$$

holds. In particular

$$\text{Regular} \implies \text{Patch trivial}$$

is a restatement of Theorem 7.1.4. There is a lot of slack in the proof of that result, and we will show that regularity is much stronger than required. We do this by an analysis of how v_F is generated.

For now let's try to get a feel for the point-sensitive nature of patch triviality.

Suppose the space S is sober and packed. Thus by Lemma 4.1.2 it is also T_1. The patch assembly diagram of Section 7.3 gives a pair of morphisms

$$\mathcal{O}S \underset{\pi_S}{\overset{}{\rightleftarrows}} P\mathcal{O}S$$

which compose to give the identity morphism on $\mathcal{O}S$. It is tempting to think that the other composite is also the identity (on $P\mathcal{O}S$) and hence the topology $\mathcal{O}S$ is patch trivial.

Example 8.1.3. There are spaces S which are sober and packed but where the topology $\mathcal{O}S$ is not patch trivial. A collection of these examples is discussed in Chapter 11. □

This example shows there is some kind of discrepancy between the point-sensitive patch construction and the point-free version. This will be analysed in more detail in Chapter 9. The example shows also that sober and packed are not sufficient to ensure that a topology is patch trivial. By Theorem 7.1.4 we know that T_3 is sufficient. However, T_3 is not necessary, as shown by either part of the following result.

Lemma 8.1.4. *Suppose the space S is either*
(a) T_2 or
(b) T_0 + fit + packed (hence T_1 + sober)
then $\mathcal{O}S$ is patch trivial.

Proof. Part (a) will be proved later as part of Theorem 8.4.4.

(b) Consider any v_F where F is open. This F is determined by some $Q \in \mathcal{Q}S$. On general grounds v_F and $[Q']$ are companions, so $v_F = [Q']$ by fitness. The packed property ensures that Q' is open. □

Here we have two conditions which imply patch triviality. However, neither of these is necessary.

Example 8.1.5. (a) The subregular topology on the real numbers discussed in Section 10.2 is a T_2 space (hence packed and sober) which is not fit. In particular, the condition

$$T_0 + \text{ fit } + \text{ packed}$$

is not necessary to achieve a patch trivial topology.

(b) The maximal compact topology is a

$$T_0 + \text{ fit } + \text{ packed } + \text{ compact}$$

space which is not T_2. In particular T_2 is not necessary to achieve a patch trivial topology. □

This was the haphazard collection of results we started out with. They don't seem to give much of a clue about the nature of patch triviality. In the next section, we introduce a hierarchy of properties which will organise and explain most of these results.

8.2 Stratified tidiness

Recall that in Section 5.6 when we introduced the v nucleus associated with an open filter we let f be the pointwise supremum

$$\dot{\bigvee}\{v_a \mid a \in F\}$$

and constructed a sequence of elements

$$d(0) = \bot \qquad d(\alpha+1) = fd(\alpha) \qquad d(\lambda) = \bigvee\{d(\alpha) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. We saw that this eventually stabilises at some element

$$d = d(\infty) = v_F(\bot)$$

for some ordinal ∞.

So using this notation, Question 8.1.2 asks when is it the case that

$$v_F = u_d$$

for each open filter F?

We know that $u_d \leq v_F$ on general grounds, so we only need to consider the conditions under which $v_F \leq u_d$ holds. For this, we claim that the condition

$$x \in F \Longrightarrow d \vee x = \top$$

is both necessary and sufficient. We will prove this in Lemma 8.2.2

This motivates the following definitions.

Definition 8.2.1. Let A be a frame and let α be an ordinal.

An open filter F on A is α-tidy if

$$x \in F \Longrightarrow d(\alpha) \vee x = \top$$

where $d(\alpha) = f^\alpha\bot$ (as above).

The frame A is α-tidy if each of its open filters is α-tidy.

We say A is *tidy* if it is α-tidy for some ordinal α.

If A is tidy, then its *tidiness* is the smallest ordinal α such that A is α-tidy. □

This definition attaches an ordinal measure to each frame which, as we will see, is an indication of how neat it is.

Lemma 8.2.2. *A frame is tidy if and only if it is patch trivial.*

Proof. For an open filter F, if

$$x \in F \Longrightarrow d(\alpha) \vee x = \top$$

holds, then $F \subseteq \nabla(u_d)$, giving $v_F \leq u_d$ and thus $v_F = u_d$. This shows that A is patch trivial.

Conversely, suppose

$$A \longrightarrow PA$$

is an isomorphism. Consider an open filter F. We must have $v_F = u_d$ for some $d \in A$, and this can only be

$$d = v_F \bot = d(\infty)$$

for some ordinal ∞. Then

$$F = \nabla(v_F) = \nabla(u_d)$$

and so

$$x \in F \Longrightarrow d \vee x = \top$$

holds. □

Essentially, a frame is tidy when its patches don't show. This, with the stratified version, is the better terminology promised in Section 8.1.

Trivially, if a filter or frame is α-tidy then it is β-tidy for all $\beta \geq \alpha$. Thus we have a hierarchy of properties. Of course, we have not yet shown that all these properties are distinct, but we will do so in Theorem 11.3.7 when we examine a series of examples.

Observe that

$$A \text{ is 0-tidy} \iff A \text{ is trivial (the one element frame)}$$

however 1-tidyness is much more interesting. Theorem 7.1.4 says

$$\text{regular} \Longrightarrow \text{tidy}$$

and this will be refined to

$$\text{regular} \Longrightarrow \text{1-tidy}$$

which, with the non-collapsing hierarchy gives a vast improvement.

We will want to use tidyness in the spatial case where the frame under consideration is a topology.

In this situation, the descending sequence of closed sets described in Section 5.6 comes in to play.

Lemma 8.2.3. *Let S be a sober space, let $Q \in \mathcal{Q}S$ and let F be the corresponding filter on $\mathcal{O}S$. For each ordinal α, the filter F is α-tidy precisely when $Q(\alpha) = Q$.*

Proof. By definition $F = \nabla(Q)$ is α-tidy precisely when for each $U \in \mathcal{O}S$

$$U \in F \Longrightarrow Q(\alpha)' \cup U = S$$

since $d(\alpha) = Q(\alpha)'$. In other words when

$$Q \subseteq U \Longrightarrow Q(\alpha) \subseteq U$$

holds. That is, when $Q = Q(\alpha)$ since Q is saturated. □

8.3 Stratified regularity

Tidyness gives us a hierarchy of separation properties for frames. In this section we show that this is closely related to another hierarchy of properties which are similar to regularity. This α-regularity comes in two closely related forms.

Definition 8.3.1. Let A be a frame and α some ordinal. We say that A is:

(a) *weak α-regular* if for each pair of elements $a \not\leq b$ and each open filter F containing a, there exist elements x, y such that

$$a \vee x = \top \quad y \leq a \quad y \not\leq b \quad x \wedge y \leq d(\alpha)$$

hold.

(b) *α-regular* if for each pair of elements $a \not\leq b$ there is an element y such that for each open filter F with $a \in F$ there is an element x such that

$$a \vee x = \top \quad y \leq a \quad y \not\leq b \quad x \wedge y \leq d(\alpha)$$

hold. □

Compare this with Definition 3.5.1. The difference between weak α-regularity and α-regularity is in the order of the quantifiers. Weak α-regular asks only that

$$(\forall a)(\forall F)(\exists y)(\exists x)\ldots$$

whereas α-regular insists that

$$(\forall a)(\exists y)(\forall F)(\exists x)\ldots$$

hold. In other words, α-regularity requires some uniformity in the choice of y.

At first weak α-regularity seems the obvious one to use, and was in fact the first we tried. However, the second notion ties in better with some of the other properties we expect of regularity, in particular the α-indexed version of the well-inside relation defined below.

In Lemma 3.5.4 we saw that frame is regular if and only if every element is the join of elements well-inside it. There is a corresponding notion of α-well-inside to go with α-regularity.

Definition 8.3.2. We say that y is α-well-inside a (and write $y \lessdot_\alpha a$) if for every open filter F such that $a \in F$ there exists x such that

$$a \vee x = \top \qquad y \leq a \qquad x \wedge y \leq d(\alpha)$$

hold. □

Notice that $\lessdot_0$ is just the usual definition of $\lessdot$ we saw in Definition 3.5.3. From the definitions of α-regular and α-well-inside we see that A is α-regular exactly when for each pair $a \not\leq b$ there is some y such that

$$y \lessdot_\alpha a \qquad y \not\leq b$$

hold.

Lemma 8.3.3. *A frame A is α-regular if and only if every element of A is the join of elements α-well-inside it.*

Proof. Suppose A is α-regular. For $a \in A$ let

$$b = \bigvee\{y \mid y \lessdot_\alpha a\}$$

so that $b \leq a$. If $a \not\leq b$ then by the definition of α-regularity

$$y \lessdot_\alpha a \qquad y \not\leq b$$

for some $y \in A$ which is a contradiction.

Conversely, suppose

$$a = \bigvee\{y \mid y \lessdot_\alpha a\}$$

for each $a \in A$. Then

$$a \not\leq b \Longrightarrow (\exists y)[y \lessdot_\alpha a \text{ and } y \not\leq b]$$

which verifies α-regularity. □

The following properties of α-regularity are immediate from the definition.

Lemma 8.3.4. *For each frame A and ordinals $\alpha \leq \beta$ the following implications hold.*
(a) α-regular $\Longrightarrow$ weak α-regular.
(b) α-regular $\Longrightarrow$ β-regular.
(c) Weakly α-regular $\Longrightarrow$ weakly β-regular.
(d) 0-regular $\Longleftrightarrow$ regular.

We now have three hierarchies of properties. In fact, these interlace.

Theorem 8.3.5. *For each frame A the implications*

$$\alpha\text{-tidy} \Longrightarrow \alpha\text{-regular} \Longrightarrow \text{weakly } \alpha\text{-regular} \Longrightarrow (\alpha+1)\text{-tidy}$$

hold for each ordinal α.

Proof. Suppose A is α-tidy. Consider elements $a \not\leq b$ of A. Let $y = a$, and suppose that F is an open filter with $a \in F$. Then α-tidyness gives $a \vee d(\alpha) = \top$. Setting

$$x = a \supset d(\alpha) \qquad y = a$$

gives us

$$x \wedge y = x \wedge a \leq d(\alpha) \qquad a \vee x \geq a \vee d(\alpha) = \top$$

and $y \leq a$ with $y \not\leq b$ to verify α-regularity.

It is trivial to see that α-regularity implies weak α-regularity.

Suppose A is weak α-regular. Consider any open filter F and $a \in F$. The weak α-regularity gives

$$a = \bigvee\{y \mid y \leq a \text{ and } a \vee (y \supset d(\alpha)) = \top\}$$

and this supremum is directed. Since $a \in F$, this shows that

$$a \vee (y \supset d(\alpha)) = \top$$

for some $y \leq a$ with $y \in F$. But now

$$y \supset d(\alpha) = v_y d(\alpha) \leq f d(\alpha) = d(\alpha + 1)$$

and therefore A is $(\alpha + 1)$-tidy. □

Remembering Lemma 8.3.4(c), a particular case of Theorem 8.3.5 is

$$\text{regular} = \text{0-regular} \Longrightarrow \text{1-tidy}$$

which is our version of Theorem 7.1.4.

8.4 The spatial case

On a spatial frame, how do these properties relate to the established separation properties, especially T_0, T_1, T_2 and T_3?

Lemma 8.4.1. *If the frame A is tidy then each point of A is maximal in A and the point space is T_1.*

Proof. Let S be the point space of A. Consider any $p \in S$ and let P be the corresponding completely prime filter, that is

$$y \in P \Longleftrightarrow y \nleq p$$

for $y \in A$. Notice that v_P is a patch nucleus since P is open. Let

$$d = v_P\bot \leq w_p\bot = p$$

and consider any $a > p$. Since $a \nleq p$ we have $a \in P$ and thus tidyness gives

$$a = a \vee p \geq a \vee d = \top$$

therefore p is maximal. □

We can extend this result to T_0 spaces in general. Each T_0 space is a subspace of its sober reflection ${}^+S$, and the two have isomorphic topologies. If the frame $\mathcal{O}S$ is tidy, then by the previous lemma its point space ${}^+S$ is T_1. Lemma 2.1.4 states that if a space S has a sober reflection which is T_1, then S is already T_1 and sober. This gives the following result.

Lemma 8.4.2. *If a T_0 space has a tidy topology then it is T_1 and sober.*

A T_0 space is T_3 precisely when it is 0-regular. What about the next level of the interlacing hierarchy? To answer that we refine Lemma 8.4.1

Lemma 8.4.3. *If a frame A is 1-tidy then its point space S is T_2.*

Proof. Suppose $p \in S$ with corresponding filter P. Then

$$d(1) = \bigvee\{\neg x \mid x \nleq p\}$$

and because A is 1-tidy,

$$a \nleq p \Longrightarrow a \vee d(1) = \top$$

for $a \in A$. Consider any point $q \neq p$. We need to find disjoint open neighbourhoods for p and q. We know by the previous lemma that points are maximal, hence $q \nleq p$ in A, and hence $q \in P$ and so $q \vee d(1) = \top$, to give $d(1) \nleq q$. Thus there exists some $x \nleq p$ with $y = \neg x \nleq q$ and hence we have

$$p \in U(x) \qquad q \in U(y) \qquad U(x) \cap U(y) = U(\bot) = \varnothing$$

which is a T_2 separation of p and q as required. □

Theorem 8.4.4. *A T_0 space S has a 1-tidy topology if and only if S is T_2.*

Proof. Each T_0 space is a subspace of its sober reflection. If such a space has a 1-tidy topology, then by the previous lemma it is a subspace of a T_2 space, and hence is T_2 itself.

We now just need to show that if S is T_2 then its topology is 1-tidy. Suppose F is an open filter on the frame $\mathcal{O}S$. Since S is T_2 and therefore sober, we have $F = \nabla(Q)$ for some compact saturated subset Q of S. In other words,

$$U \in F \iff Q \subseteq U$$

for $U \in \mathcal{O}S$. As usual

$$Q(1) = \bigcup\{U^- \mid Q \subseteq U\}$$

and so $Q \subseteq Q(1)$.

It remains to show that $Q(1) \subseteq Q$. Suppose $p \notin Q$. By Corollary 2.2.12 there exist open sets U, V such that

$$p \in U \qquad Q \subseteq V \qquad U \cap V = \varnothing$$

hold. Thus $p \notin V^-$ and so $p \notin Q(1)$. This proves the result. □

There are several attempted characterisations of Hausdorff for frames. Here we have another one. We don't at present know how it relates to the others.

To conclude this section let's gather together the various characterisations we have obtained so far, and one we will obtain in the next section.

For each T_0 space we have the following.

(Page 90)	$\mathcal{O}S$ is 0-tidy	$\iff$	$S = \varnothing$
(Lemma 8.3.4(d))	$\mathcal{O}S$ is 0-regular	$\iff$	S is T_3
(Theorem 8.4.4)	$\mathcal{O}S$ is 1-tidy	$\iff$	S is T_2
	$\mathcal{O}S$ is 1-regular	$\iff$	??
(Theorem 8.5.12)	$\mathcal{O}S$ is tidy	$\iff$	S is packed + stacked

Stacked spaces are the topic of the next section.

8.5 Stacked spaces

Recall the following two observations.

- A frame A is patch trivial exactly when it is tidy (Lemma 8.2.2). This enables us to associate a rank, the degree of tidyness, to such a frame.
- For a topological space S, we have ${}^{p}S = S$ exactly when S is packed (Section 4.1).

What is the relationship between these two notions? In this section we show that the topology of a T_0 space is tidy exactly when the space is packed in a rather neat fashion. This enables us to make a perhaps unexpected connection with another topic that will be discussed in Section 8.6.

Lemma 8.5.1. *Let A be a frame with point space S. If A is tidy then S is packed.*

Proof. Consider $Q \in \mathcal{Q}S$ and let $F = \nabla(Q)$ be the associated filter on A. Recall from Section 6.4 that $\Sigma_A = \sigma_{\mathcal{O}S} \circ NU_A$. We know $\Sigma(v_F) = Q'$. When A is tidy we have $v_F = u_d$ for some $d \in A$, and hence

$$Q' = \Sigma(v_F) = \Sigma(u_d) = U(d)$$

to show Q is closed. □

As a particular case of this, if a sober space has a tidy topology then it is packed. This looks as though it might be an equivalence but it is not. The following result was mentioned earlier in Section 8.1, now we give it a name.

Lemma 8.5.2. *If the space S is sober and packed then the canonical patch assembly embedding is split*

$$\mathcal{O}S \rightleftarrows P\mathcal{O}S$$

that is it has a one-sided inverse where the composite on $\mathcal{O}S$ is the identity.

The proof follows directly from looking at the full patch assembly diagram from Section 7 in the case where the insertion

$$\mathcal{O}S \longrightarrow \mathcal{O}^p S$$

is an isomorphism.

Lemma 8.5.1 shows that if the topology of a sober space S is tidy then S is packed. However the examples in Chapter 11 shows that sober and packed are not enough to achieve tidyness. We need something extra. To analyse this we use a relation on the subsets of a space.

Definition 8.5.3. Let S be a space and let $Q \in \mathcal{Q}S$. We say a closed set $X \in \mathcal{C}S$ is Q-irreducible and write

$$Q \ltimes X$$

if

$$Q \subseteq U \Longrightarrow X \subseteq (X \cap U)^-$$

holds for each $U \in \mathcal{O}S$. □

Since the inclusion $(X \cap U)^- \subseteq X$ is trivial, the relationship $Q \ltimes X$ can be rephrased as

$$Q \subseteq U \Longrightarrow X = (X \cap U)^-$$

(for $U \in \mathcal{O}S$). This strengthening will be useful in some calculations.

What has this notion, Q-irreducibility, got to do with the more standard notion of irreducibility? For each point x of a space the saturation $\uparrow x$ is compact. We look at $(\uparrow x)$-irreducibility.

Lemma 8.5.4. *Let S be a space and let X be a non-empty closed set. Then X is irreducible exactly when*

$$x \in X \Longrightarrow (\uparrow x) \ltimes X$$

for each $x \in S$.

Proof. Suppose first that X is irreducible, consider any $x \in X$, and any
$U \in \mathcal{O}S$ with $x \in U$. Let

$$V = (X \cap U)^{-\prime}$$

so we require $X \subseteq V'$, that is $X \cap V = \varnothing$. By way of contradiction suppose X meets V. We know X meets U (at x), so the irreducibility gives

$$\varnothing \neq X \cap U \cap V \subseteq V' \cap V = \varnothing$$

which is nonsense.

Secondly suppose

$$x \in X \Longrightarrow (\uparrow x) \ltimes X$$

for each $x \in S$. Suppose $U, V \in \mathcal{O}S$ and

$$x \in X \cap U \qquad y \in X \cap V$$

say. We must show that $X \cap U \cap V$ is non-empty. We have

$$(X \cap U)^{-} = X = (X \cap V)^{-}$$

by the assumed property of X. In particular

$$y \in (X \cap U)^{-}$$

(since $y \in X$). But $y \in V \in \mathcal{O}S$, and hence

$$X \cap U \cap V \neq \varnothing$$

as required. □

We use the relation $\ltimes$ to impose two conditions on a space.

Definition 8.5.5. A space S is *stacked* if

$$Q \ltimes X \Longrightarrow X \subseteq Q^{-}$$

holds for each $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$.

A space S is *strongly stacked* if

$$Q \ltimes X \Longrightarrow X \subseteq (X \cap Q)^{-}$$

holds for each $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. □

We will analyse the connections between these various notions and show how they relate to tidyness. First we look at the straightforward connections.

Lemma 8.5.6. *(a) Each T_2 space is strongly stacked.*

(b) Each strongly stacked space is stacked.

(c) Each stacked T_1 space is sober.

Proof. (a) Suppose S is a T_2 space, and suppose $Q \ltimes X$ where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. It is sufficient to show $X \subseteq Q$.

By way of contradiction suppose $X \nsubseteq Q$. Thus there is a point $p \in X - Q$. Since S is T_2, Corollary 2.2.12 gives

$$Q \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

for some open sets U and V. Since $Q \ltimes X$ this gives

$$X \subseteq (X \cap U)^- \subseteq U^- \subseteq V'$$

and hence

$$p \in V \cap X \subseteq V \cap V' = \varnothing$$

which is nonsense.

(b) This is trivial.

(c) Suppose the space S is T_1 and stacked. Consider any closed irreducible set $X \subseteq S$ and consider any $x \in X$. We show $X = \{x\}$.

By Lemma 8.5.4 we have $(\uparrow x) \ltimes X$. Since S is T_1 we have

$$(\uparrow x) = \{x\} = x^-$$

and since S is stacked we have

$$x \in X \subseteq (\uparrow x)^- = \{x\}$$

as required. □

There is a lot of slack in part (a) of this result. Most spaces we want to look at appear to be strongly stacked; not many are T_2. The following result illustrates that there are many more strongly stacked spaces out there.

Lemma 8.5.7. *Every Alexandroff topology is strongly stacked.*

Proof. For $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$ such that $Q \ltimes X$ we have

$$Q \subseteq U \Longrightarrow X \subseteq (X \cap U)^-$$

by definition. But in an Alexandroff space every saturated set is open, so a special case of the above condition gives

$$Q \ltimes X \Longrightarrow X \subseteq (X \cap Q)^-$$

which is precisely the definition for a space to be strongly stacked. □

At first sight the point-sensitive relation $\ltimes$ looks a bit mysterious. Its role in life is partly explained by Lemma 8.5.4. Its true purpose becomes clear when we look at the point-free situation.

Each $Q \in \mathcal{Q}S$ gives an open filter F on $\mathcal{O}S$ which, in turns, induces an inflator $f = f_F$ and a nucleus $v_F = f^\infty$ on $\mathcal{O}S$. We also have the spatially induced nucleus $[Q']$ on $\mathcal{O}S$, and we know $v_F \leq [Q']$ (since F is the admissible filter of both).

We have

$$f(W) = \bigcup\{(U \supset W)^\circ \mid Q \subseteq U\}$$

for each $W \in \mathcal{O}S$. Thus, for each $X \in \mathcal{C}S$ we have

$$\begin{aligned} f(X') &= \bigcup\{(U' \cup X')^\circ \mid Q \subseteq U\} \\ &= \bigcup\{(X \cap U)^{-\prime} \mid Q \subseteq U\} \\ &= \Big(\bigcap\{(X \cap U)^- \mid Q \subseteq U\}\Big)' \\ &= \Big(\widehat{Q}(X)\Big)' \end{aligned}$$

where $\widehat{Q}$ is as we defined in Section 5.6. This gives the following.

Lemma 8.5.8. *For each space S and $Q \in \mathcal{Q}S$ we have*

$$Q \ltimes X \iff \widehat{Q}(X) = X \iff v_F(X') = X'$$

for each $X \in \mathcal{C}S$.

Lemma 8.5.9. *For each space S and $Q \in \mathcal{Q}S$ we have*
(a) $Q^- \subseteq Q(\infty)$
(b) $Q \ltimes Q(\infty)$
(c) $Q \ltimes X \Longrightarrow X \subseteq Q(\infty)$
for each $X \in \mathcal{C}S$.

Proof. (a) We have $Q \subseteq Q(\infty)$ by construction, and $Q(\infty)$ is closed therefore the result follows.

(b) By definition

$$Q(\infty) = \widehat{Q}(Q(\infty)) = \bigcap\{(Q(\infty) \cap U)^- \mid Q \subseteq U\}$$

so

$$Q \subseteq U \Longrightarrow Q(\infty) \subseteq (Q(\infty) \cap U)^-$$

as required.

(c) By construction $Q(\infty)$ is the largest set Y with $\widehat{Q}(Y) = Y$. □

We have seen that

$$Q \subseteq Q^- \subseteq Q(\infty)$$

for each $Q \in \mathcal{Q}S$. By definition, the left hand inclusion is an equality precisely when the space S is packed. We have a similar explanation of being stacked.

Corollary 8.5.10. *A space S is stacked precisely when $Q^- = Q(\infty)$ for each $Q \in \mathcal{Q}S$.*

Proof. Since $Q \ltimes Q(\infty)$ by Lemma 8.5.9, stacked implies that

$$Q(\infty) \subseteq Q^-$$

to give the inclusion we need. □

Strongly stacked spaces go one step further.

Lemma 8.5.11. *For each space S the following are equivalent.*

(a) S is strongly stacked.

(b) For each open filter F on $\mathcal{O}S$ we have $v_F = [Q']$ where F is the neighbourhood filter of $Q \in \mathcal{Q}S$.

(c) For each open filter F on $\mathcal{O}S$ the nucleus v_F is spatially induced.

Proof. (a)$\Longrightarrow$ (b). Suppose S is strongly stacked. Consider $Q \in \mathcal{Q}S$ and let F be the corresponding open filter on $\mathcal{O}S$. We saw in Section 5.7 that $v_F \le [Q']$, so it suffices to show $[Q'] \le v_F$. For each $X \in \mathcal{C}S$ we have

$$\begin{aligned} v_F(X') = X' &\Longrightarrow Q \ltimes X \\ &\Longrightarrow X \subseteq (X \cap Q)^- \\ &\Longrightarrow [Q'](X') = X' \end{aligned}$$

by Lemma 8.5.8, the strongly stacked property of S and the definition of $[Q']$. This shows that any open set fixed by v_F is also fixed by $[Q']$, and hence $[Q'] \le v_F$ as required.

(b)$\Longrightarrow$ (c) is trivial.

(c)$\Longrightarrow$ (a). Suppose $v_F = [E']$ for some $E \subseteq S$. Then

$$Q \subseteq U \Longleftrightarrow E \subseteq U$$

for $U \in \mathcal{O}S$. Thus $E \subseteq Q$ (since Q is saturated). This gives

$$[Q'] \le [E'] = v_F$$

as required. □

With this we can bring several results together to obtain the characterisation of point-sensitive tidyness.

Theorem 8.5.12. *A T_0 space S has a tidy topology precisely when it is both packed and stacked.*

Proof. Suppose first that $\mathcal{O}S$ is tidy and consider $Q \in \mathcal{Q}S$. We have

$$Q \subseteq Q^- \subseteq Q(\infty)$$

so it suffices to show $Q(\infty) \subseteq Q$. Let F be the filter on $\mathcal{O}S$ induced by Q. Since $\mathcal{O}S$ is tidy we have $v_F = [D]$ for some $D \in \mathcal{O}S$. In fact $D = Q(\infty)'$. But now

$$Q \subseteq U \Longleftrightarrow v_F(U) = S \Longleftrightarrow D \cup U = S \Longleftrightarrow Q(\infty) \subseteq U$$

for each $U \in \mathcal{O}S$. Since Q is saturated, this gives the required result.

Conversely, suppose S is packed and stacked, and consider any open filter F on $\mathcal{O}S$. By Lemma 4.1.2 the space S is T_1 and hence S is sober by Lemma 8.5.6 (c). We know that F is induced by some $Q \in \mathcal{Q}S$. In general we have

$$[Q(\infty)'] \le v_F \le [Q']$$

but in a packed and stacked space $Q(\infty) = Q \in \mathcal{C}S$. this says that $v_F = [D]$ for some $D \in \mathcal{O}S$ and so $\mathcal{O}S$ is tidy. □

8.6 Vietoris points

In this section we see how the material of this chapter has an impact on another topic, the vietoris points of a frame. A detailed account of this is given in [15]. Here I will describe the main connections without proofs. These results are not mine and are included here merely to round out the picture.

The original construction (due to Vietoris and others) starts with a fairly nice space S and produces a second space where the points are certain subsets of the first. On reflection it seems that $\mathcal{Q}S$ is the most obvious set on which to impose the second topology. (In the original construction S is compact T_2 and then $\mathcal{Q}S = \mathcal{C}S$.) Because of later developments it turns out that the family of compact lenses is a better carrier of the constructed space. A compact lens is a subset

$$L = Q \cap X$$

for $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$.

A point-free version of this construction is given in [10], with a special case described in [9]. This version starts with an arbitrary frame A to produce a second frame VA, the V-modification of A. Even when A is spatial, the frame VA need not be, but it does have a point space $\mathsf{pt}(VA)$. The topology on $\mathsf{pt}(VA)$ is canonical.

When A is spatial, $A = \mathcal{O}S$, this gives us (at least) two modification spaces, the space $\mathsf{pt}(VA)$ and the space (or spaces) obtained by the point-sensitive construction. What is the relationship between these spaces? When S is 'nice' they are the same, but in general they are different. Here being 'nice' is related to the stacking properties of S.

What are the V-points of a frame A, that is the points of VA? We have had a glimpse of these in Section 5.7. The following result is Theorem 5.7 of [15].

Theorem 8.6.1. *For each frame A the V-points are the pairs (F, a) where F is an open filter on A and a is a member of the principal lower section*

$$[v_F(\bot), w_F(\bot)]$$

of A_F.

We now see that Section 5.7 is actually about V-points. As indicated there the space $\mathsf{pt}(VA)$ can be quite complicated.

When A is spatial, that is when $A = \mathcal{O}S$ for a sober space S, we can describe these V-points (F, a) directly in terms of S.

The first component F is an open filter on $\mathcal{O}S$ and so corresponds to some $Q \in \mathcal{Q}S$. Recall also that Q is the saturation of its set M of minimal members, and sits inside $Q(\infty)$. The second component is an open set, but in this setting it is more convenient to deal with the complement.

The following version of Theorem 8.6.1 is given in Section 6 of [15].

Theorem 8.6.2. *For each sober space S the V-points of $\mathcal{O}S$ are the pairs (Q, X) where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$ with both $Q \ltimes X$ and $M^- \subseteq X \subseteq Q(\infty)$.*

For a fixed $Q \in \mathcal{Q}S$, which closed sets $X \in \mathcal{C}S$ give a V-point (Q, X)? We have

$$M^- \subseteq Q^- \subseteq Q(\infty)$$

and each of these three sets gives a V-point. In fact, M^- and $Q(\infty)$ are the two extremes and, depending on the space, Q^- can float about within this interval. Of course, these closed sets need not be distinct. In fact, when the topology $\mathcal{O}S$ is tidy we have $M = Q(\infty)$ and the set of V-points of $\mathcal{O}S$ is essentially $\mathcal{Q}S$. It turns out that in this case the canonical topology on $\mathsf{pt}(V\mathcal{O}S)$ is exactly that given by the original point-sensitive construction.

What is $\mathsf{pt}(V\mathcal{O}S)$ when $\mathcal{O}S$ is not tidy?

Recall that a compact lens has the form

$$L = Q \cap X$$

where $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$. For such a lens L we have $M \subseteq L \subseteq Q$, and hence Q is the saturation of L. Also $L^- \subseteq X$, but these sets can be different. Nevertheless we do have

$$L = Q \cap L^-$$

that is we can also replace X by L^-.

For each compact lens L as above the pair (Q, L^-) is a V-point. These are the *focal points* of $\mathcal{O}S$. It is simple to show that a V-point (Q, X) is a focal point precisely when $X = (Q \cap X)^-$. In particular, for a fixed $Q \in \mathcal{Q}S$ the two sets M^- and Q^- provide focal points, and these are the extreme focal points.

With a bit more work the following can be obtained.

Theorem 8.6.3. *Let S be a sober space. If S is strongly stacked then each V-point of $\mathcal{O}S$ is a focal point.*

If each V-point of $\mathcal{O}S$ is a focal point then S is stacked.

At the time of writing the precise relationship between these three properties is not known. What is known is that the set of focal points can be viewed as a subspace of $\mathsf{pt}(V\mathcal{O}S)$ with the topology given by the original point-sensitive construction.

These results indicate that when S is 'nice' the point-free construction reproduces one of the point-sensitive constructions. This can be made precise as follows.

Theorem 8.6.4. *Let S be a sober space.*

(a) The space S is T_1 precisely when $M^- = Q^-$ for each $Q \in \mathcal{Q}S$, that is when each $Q \in \mathcal{Q}S$ has precisly one associated focal point.

(b) The space S is T_1 and stacked precisely when $M^- = Q(\infty)$ for each $Q \in \mathcal{Q}S$, that is when each $Q \in \mathcal{Q}S$ has precisely one associated V-point.

These results indicate that the stacking properties of a sober space have something to do with the relationship between the point-free and the point-sensitive V-constructions. Given this we can look at Example 5.7.3 in a different way.

Example 8.6.5. There is a space S which is T_1, sober and tightly packed. The space S has a special point $*$ and for $Q = \{*\}$ there is just one associated focal point (since $M = Q = \{*\}$) but there is a veritable forest of associated V-points. □

As indicated in Section 5.7 the deleted set $\mathbb{S} = S - \{*\}$ is a 'large' tree with many 'large' subtrees. Each such subtree gives an associated V-point. The details of this are given in Chapter 11 where a collection of similar examples is discussed.

Chapter 9

The point space of the patch assembly

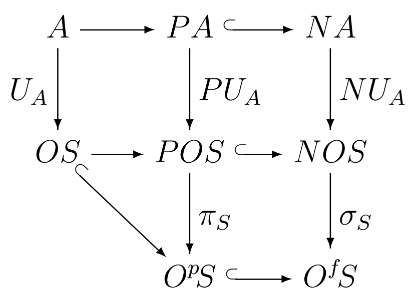
For a frame A with point space S there are two patch related frames

$$\mathcal{O}\mathsf{pt}(PA) \qquad P\mathcal{O}S = P(\mathcal{O}\mathsf{pt}A)$$

the topology of the point space of the patch assembly and the patch assembly of the point space topology. In this chapter we investigate the connection between these and certain related spaces.

9.1 The basic information

In Section 7.3 we constructed the following commuting diagram giving the relationship between the patch and full assemblies on a frame A and the patch and front topologies on its point space S.



Let's set down what we know about this diagram.

• Each horizontal arrow is an embedding and, as indicated, three of them are inclusions.

• By construction the spatial reflection arrow U_A is surjective.

• The functorial properties of N ensure that both NU_A and PU_A are surjective.

• The arrow σ_S is surjective. Furthermore, the space ${}^fS$ is the point space of both $N\mathcal{O}S$ and NA, where σ_S and the composite $\Sigma_S = \sigma_S \circ NU_A$ are the spatial reflection arrows.

• The arrow π_S is surjective. Indeed, for each $Q \in \mathcal{Q}S$ we have

$$\pi(v_F) = Q'$$

where F is the open filter on $\mathcal{O}S$ generated by Q.

This information has an obvious omission which prompts some questions.

- What is the point space $\mathsf{pt}(PA)$ of the patch assembly PA of A?
- In particular, what is the point space of $P\mathcal{O}S$?
- Are these spaces different?

In this chapter we investigate these and related questions. At the time of writing we do not have all the answers, but we do have some useful information.

9.2 Two spoilers

In a way the front space ${}^fS$ of S is a rather crude version of the patch space ${}^pS$. Nevertheless ${}^fS$ is the point space of both $N\mathcal{O}S$ and NA. Perhaps, in a similar way, ${}^pS$ is the point space of $P\mathcal{O}S$ or PA or both. Let's nail that one straight away.

Example 9.2.1. The patch space ${}^pS$ of a sober space S need not be sober. For instance, this is the case when S is the sober reflection of the cocountable topology; see Section 10.1. For such a space S the patch space ${}^pS$ can not be the point space of anything. $\square$

Of course, there are cases where ${}^pS$ is the point space of $P\mathcal{O}S$. For instance, if S is T_2 then ${}^pS = S$ and

$$\mathcal{O}S \longrightarrow P\mathcal{O}S$$

is an isomorphism. However, in general we have to search a bit harder to find the point space.

Perhaps things aren't too bad and we can show that $P\mathcal{O}S$ is always spatial. That is easily nailed as well.

Example 9.2.2. There is a sober space S such that $P\mathcal{O}S$ is not spatial. See Chapter 11 for a collection of such examples. This result is proved as Theorem 11.3.5. $\square$

These two examples suggest that we still have a bit of work to do, so let's conclude this section on a positive note.

Lemma 9.2.3. *For a sober space S, if the canonical surjection*

$$P\mathcal{O}S \xrightarrow{\ \pi\ } \mathcal{O}{}^pS$$

is an isomorphism, then S is strongly stacked.

Proof. Consider $Q \in \mathcal{Q}S$. By Lemma 8.5.11 it is sufficient to show that

$$v_F = [Q']$$

where F is the open filter on $\mathcal{O}S$ generated from Q. But, by Lemma 7.3.1, we have

$$\pi(v_F) = Q' = \pi([Q'])$$

and, by assumption, π is injective, to give the required result. $\square$

The converse is untrue, however, as it is possible to have a strongly stacked space where π is not an isomorphism. Section 10.4 gives an example of this.

9.3 The 'ordinary' points of the patch assembly

The existence of the surjective morphism

$$PA \longrightarrow P\mathcal{O}S \longrightarrow \mathcal{O}^pS$$

indicates there is some connection between ${}^pS$ and the point space $\mathsf{pt}(PA)$. In particular there must be a continuous map

$${}^pS \longrightarrow \mathsf{pt}(\mathcal{O}^pS) \longrightarrow \mathsf{pt}(PA)$$

where the central space is just the sober reflection of ${}^pS$. In this section we obtain an explicit description of this map, and we show that it exhibits ${}^pS$ as a subspace of $\mathsf{pt}(PA)$.

Recall that we view the points of a frame as its $\wedge$-irreducible elements. In particular the points of PA are those patch nuclei that are $\wedge$-irreducible as elements of PA. Recall also that the points of the full assembly NA are precisely the nuclei w_p for $p \in S$. If any such nuclei belongs to PA then it is automatically a point of PA. We show they all belong to PA.

Let p be some arbitrary point of S. Recall that

$$w_p(x) = \begin{cases} \top & \text{if } x \not\leq p \\ p & \text{if } x \leq p \end{cases}$$

for $x \in A$. Let

$$P = \nabla(w_p) = \{x \in A \mid x \not\leq p\}$$

be the associated completely prime filter, the admitting filter of w_p. This is an open filter. By Lemma 5.5.8 we know that w_p is the greatest member of the corresponding block. What is the least member v_P? To describe that we use the inflator

$$f_p = f_P = \dot{\bigvee}\{v_y \mid y \in P\}$$

so that

$$f_p(x) = \begin{cases} \top & \text{if } x \not\leq p \\ \leq p & \text{if } x \leq p \end{cases}$$

for $x \in A$. (Later we will see that $f_p(\bot) = \bot \neq p$ can happen.)

Lemma 9.3.1. *For the situation above we have*

$$w_p = u_p \vee v_P = f_p \circ u_p$$

and $w_p \in \mathsf{pt}(PA)$.

Proof. We have

$$w_p \geq u_p \vee v_P = v_P \circ u_p \geq f_p \circ u_p$$

and the description of f_p gives

$$(f_p \circ u_p)(x) = f_p(p \vee x) = \begin{cases} \top & \text{if } x \not\leq p \\ p & \text{if } x \leq p \end{cases} = w_p(x)$$

(for $x \in A$) as required.

Both u_p and v_P belong to PA, hence so does w_p. But w_p is $\wedge$-irreducible in NA, hence also in PA, so that $w_p \in \mathsf{pt}(PA)$. □

This result gives us a set insertion

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & \mathsf{pt}(PA) \\ p & \longmapsto & w_p \end{array}$$

and so imposes a topology on the set S using the given topology on $\mathsf{pt}(PA)$. To describe the imposed topology we use the canonical subbasic open sets

$$U_{PA}(u_a) \qquad U_{PA}(v_F)$$

of $\mathsf{pt}(PA)$. Here a is an arbitrary element of A and F is an arbitrary open filter on A. Recall that $Q = S - F$ is in $\mathcal{Q}S$ and determines F by

$$x \in F \Longleftrightarrow Q \subseteq U_A(x)$$

(for $x \in A$).

Lemma 9.3.2. *For the situation above we have*

$$w_p \in U_{PA}(u_a) \Longleftrightarrow p \in U_A(a) \qquad w_p \in U_{PA}(v_F) \Longleftrightarrow p \in Q'$$

for each $a \in A$, open filter F and $p \in S$.

Proof. We have

$$w_p \in U_{PA}(u_a) \Longleftrightarrow u_a \not\leq w_p \Longleftrightarrow a \not\leq p \Longleftrightarrow p \in U_A(a)$$

to give the left hand equivalence.

Remembering that v_F is a fitted nucleus we have

$$w_p \in U_{PA}(v_F) \Longleftrightarrow v_F \not\leq w_p \Longleftrightarrow F \not\subseteq \nabla(w_p) \Longleftrightarrow p \in F \Longleftrightarrow p \in Q'$$

to give the right hand equivalence. □

This result shows that α is a continuous map when S carries the patch topology.

Theorem 9.3.3. *For each frame A with point space S the insertion*

$${}^pS \longrightarrow \mathsf{pt}(PA)$$

exhibits ${}^pS$ as a subspace of $\mathsf{pt}(PA)$.

This result locates what we hope is a large part of $\mathsf{pt}(PA)$.

Definition 9.3.4. A point of PA which is not of the form w_p for some $p \in \mathsf{pt}(A)$ is a *wild point.* □

Since $\mathsf{pt}(PA)$ is sober but ${}^pS$ need not be, we know that wild points exist for some frames A. In the next section we try to capture some of these beasts.

Question 9.3.5. *Is $\mathsf{pt}(PA)$ just the sober reflection of ${}^pS$?*

Certainly the sober reflection of ${}^pS$ must sit inside $\mathsf{pt}(PA)$. By Corollary 2.3.6 is it just the front closure of ${}^pS$ in $\mathsf{pt}(PA)$. The problem is whether it is the whole of $\mathsf{pt}(PA)$. I have been unable to answer this question. At the moment I tend to the opinion that the answer is yes.

9.4 The wild points of the patch assembly

We start by giving an example of a wild point.

Example 9.4.1. The patch assembly of the sober reflection of the cocountable topology contains a wild point. See Section 10.1 for details. □

Each wild point is attached to one of the w_p points in a canonical way.

Lemma 9.4.2. *Let A be a frame with point space S and patch assembly PA. For every point $m \in \mathsf{pt}(PA)$, the element $p = m\bot$ is a point of A and is the unique $a \in A$ with $u_a \le m \le w_a$.*

Proof. The nucleus $m \in PA$ is $\wedge$-irreducible (in PA). In particular, it is not the top of PA and so $p \neq \top$. Consider $x, y \in A$ with $x \wedge y \le p$. Both u_x, u_y are in PA and

$$u_x \wedge u_y = u_{x \wedge y} \le m$$

so that one of

$$u_x \le m \qquad u_y \le m$$

holds, giving one of

$$x = u_x(\bot) \le m(\bot) = p \qquad y = u_y(\bot) \le m(\bot) = p$$

which shows that $p \in S$.

By construction we have $u_p \le m \le w_p$. Consider any $a \in A$ with $u_a \le m \le w_a$. By evaluation at $\bot$ we have

$$a = u_a(\bot) \le m(\bot) = p \le w_a(\bot) = a$$

to give $a = p$. □

This shows that whatever the points of PA are, each one has a parent p which is a point of A and the image of a point of PA.

What is it about a point of a frame that enables it to be associated with wild patch points? Recall that each maximal element of a frame is automatically a point, but there can be non-maximal points.

Lemma 9.4.3. *If the point p of the frame A is maximal, then $u_p = w_p$ and p has no associated wild points.*

Proof. The maximality of p gives

$$u_p = \begin{cases} \top & \text{if } x \not\le p \\ p & \text{if } x \le p \end{cases}$$

(for $x \in A$), so that $u_p = w_p$ and there can be nothing between these. □

We know several conditions on a frame that ensure that all points are maximal. For instance, this is the case when the frame is fit or when it is ∞-tidy. For such a frame the previous lemma tells us that the patch situation is simple.

Theorem 9.4.4. *If each point of the frame A is maximal then A has no wild points and the two spaces ${}^{p}(\mathsf{pt}A)$ and $\mathsf{pt}(PA)$ are essentially the same.*

Of course, Lemma 9.4.3 does not say that a non-maximal point must have an associated wild point. In fact, as we will see, it is not at all clear what allows or prevents the existence of wild points.

We know that for a T_1 space every point is maximal. Thus we have the following result.

Corollary 9.4.5. *If A is a frame with a T_1 point space then A has no wild points and ${}^{p}(\mathsf{pt}A) \cong \mathsf{pt}(PA)$*

What can we say about the points in $\mathsf{pt}(PA)$? Let's set up a bit of notation to be used with an arbitrary $m \in \mathsf{pt}(PA)$ and derive a few properties. Of course, if m is not wild then almost everything we do is already known.

For $m \in \mathsf{pt}(PA)$ let $p = m(\bot)$ be the associated point, and let $M = \nabla(m)$ be its admissible filter. In general this need not be open. Whatever it is, M has an associated minimum nucleus v_M, the minimum companion of m. As yet we do not know that $v_M \in PA$. Let $\mathcal{M}$ be the set of all open filters F with $F \subseteq M$. Thus $\mathcal{M}$ could be empty. Let

$$K = \bigvee \mathcal{M}$$

where this supremum is taken in the poset of all filters on A. Since

$$v_K = \bigvee \{v_F \mid F \in \mathcal{M}\}$$

we see that

$$v_K \leq v_M \leq m \leq w_p$$

and $v_K \in PA$.

Lemma 9.4.6. *Using the notation above, for each frame A and $m \in \mathsf{pt}(PA)$ we have*

$$m = u_p \vee v_M = u_p \vee v_K$$

and

$$G \cap H \subseteq M \Longrightarrow G \subseteq M \text{ or } H \subseteq M$$

for all open filters G, H.

Proof. For convenience let $k = u_p \vee v_K$ so that

$$k \leq u_p \vee v_M \leq m$$

and a comparison $m \leq k$ suffices for the first part. Since $m \in PA$ it is a supremum of nuclei $u_a \wedge v_F$ for certain $a \in A$ and open filters F. For such a nucleus we have

$$u_a \wedge v_F \leq m$$

and hence, since m is $\wedge$-irreducible in PA one of

$$u_a \leq m \qquad v_F \leq m$$

holds. These give

$$a \leq p \qquad F \subseteq M$$

and hence one of

$$u_a \wedge v_F \leq u_a \leq u_p \leq k \qquad u_a \wedge v_F \leq v_F \leq v_K \leq k$$

holds, and so

$$u_a \wedge v_F \leq k$$

in all cases. In particular, $m \leq k$ since m is the supremum of the nuclei $u_a \wedge v_F$ considered.

For the second part consider the open filters G, H with $G \cap H \subseteq M$. Then

$$v_G \wedge v_H = v_{G \wedge H} \leq v_M \leq m$$

and hence one of

$$v_G \leq m \qquad v_H \leq m$$

holds, to give either $G \subseteq M$ or $H \subseteq M$ as required. □

There is much that is not known about this situation. I will conclude this chapter with what I believe is the most important open question.

Let A be an arbitrary frame with point space S. Consider the topological embedding

$$^{p}S \longrightarrow \mathsf{pt}(PA)$$

described above. We know that ^{p}S need not be sober, but $\mathsf{pt}(PA)$ is sober. The sober reflection of ^{p}S lives inside $\mathsf{pt}(PA)$ and is just the front closure of ^{p}S. This leads to the crucial open question.

(?) For a frame A, under what circumstances is the sober reflection of ^{p}S just the space $\mathsf{pt}(PA)$?

It is possible that this is always the case, but (after many sleepless nights) I have not yet been able to find a proof or a counterexample.

Chapter 10

Examples

In this and the next chapter we gather together many of the examples which have led to a greater understanding of the patch constructions. Most of these we devised as counterexamples to some conjecture or other. Eventually several were overtaken by 'better' counterexamples. However, these earlier examples are included here for they may have attributes which are useful for other purposes.

10.1 The cofinite and cocountable topologies

It seems that the crucial properties of these two examples are as follows.

- The intersection of two non-empty open sets is non-empty.
- Each superset of a non-empty open set is open.

It is possible to do the analysis that follows in more generality using just these properties, but the extra detail is not needed here. However, it is convenient to use a notation which hints at this more general analysis.

Definition 10.1.1. (a) Let S be an infinite set. The cofinite topology on S consists of the empty set together with all subsets $U \subseteq S$ such that U' is finite.

(b) Let S be an uncountable set. The cocountable topology on S consists of the empty set together with all subsets $U \subseteq S$ such that U' is countable. □

Thus in both cases we have a set S furnished with a topology of the form

$$\mathcal{O}S = \{\varnothing\} \cup \mathcal{F}S$$

where $\mathcal{F}S$ is a certain filter of subsets of S. In fact we use

$$(a) \quad \mathcal{F}S = \mathcal{P}_{\mathsf{cof}}S \qquad (b) \quad \mathcal{F}S = \mathcal{P}_{\mathsf{cc}}S$$

the filters of cofinite and cocountable subsets, respectively. Notice that in both cases the size of S ensures that $\varnothing \notin \mathcal{F}S$, so that $\mathcal{O}S \neq \mathcal{P}S$ and the space is not discrete.

For the most part we can carry out a uniform analysis, one that applies equally well to both cases, using a common notation. Of course, every now and then we need to look at the cases separately (for they do differ in some crucial ways). In this and the next subsection we look at the point-sensitive properties. The results we achieve are gathered together in Table 10.1. The notation used will be explained as we proceed.

General	(a) Cofinite	(b) Cocountable
	$\mathcal{F}S = \mathcal{P}_{\mathsf{cof}}S$	$\mathcal{F}S = \mathcal{P}_{\mathsf{cc}}S$
$\mathcal{O}S = \{\varnothing\} \cup \mathcal{F}S$	$\mathcal{O}S = \{\varnothing\} \cup \mathcal{P}_{\mathsf{cof}}S$	$\mathcal{O}S = \{\varnothing\} \cup \mathcal{P}_{\mathsf{cc}}S$
$O^{+}S = \{\varnothing\} \cup {}^{+}\mathcal{F}S$		
$\mathcal{O}^{f+}S = \mathcal{P}S \cup {}^{+}\mathcal{F}S = \mathcal{P}S \cup \mathcal{O}^{f}S$		
	$\mathcal{Q}S = \mathcal{P}S$	$\mathcal{Q}S = \mathcal{P}_{\mathsf{fin}}S$
	$\mathcal{G}S = \mathcal{P}S$	$\mathcal{G}S = \mathcal{P}_{\mathsf{cc}}S$
$\mathcal{O}^{p}S = \{\varnothing\} \cup \mathcal{G}S$	$\mathcal{O}^{p}S = \mathcal{P}S$	$\mathcal{O}^{p}S = \mathcal{O}S$
$\mathcal{O}^{p+}S = \{\varnothing\} \cup {}^{+}\mathcal{F}S \cup \mathcal{G}S = \mathcal{O}^{+}S \cup \mathcal{G}S$	$\mathcal{O}^{p+}S = \mathcal{O}^{f+}S$	$\mathcal{O}^{p+}S = \{\varnothing\} \cup {}^{+}\mathcal{F}S \cup \mathcal{F}S$

Table 10.1: Various families associated with the spaces

After that we look at the point-free properties.

Both these spaces are T_1 since each singleton is closed. However, neither is sober since $\mathcal{F}S = \mathcal{O}S - \{\varnothing\}$ is a filter, and hence the whole space S is closed and irreducible but not a point closure. Luckily the inebriation is not too bad.

Lemma 10.1.2. *Consider the spaces S of Definition 10.1.1. In each case, each proper, closed irreducible subset is a singleton.*

Proof. Let X be a closed irreducible subset with $X \neq S$. In particular $X \neq \varnothing$ (by definition of irreducibility). By way of contradiction suppose X has at least two members, say x, y. Since X is closed, both the sets

$$U_x = X' \cup \{x\} \qquad U_y = X' \cup \{y\}$$

are open (since each is a superset of an open set). Each meets X (at x and y respectively), and hence (by the irreducibility) the set

$$U_x \cap U_y = X'$$

meets X which is a contradiction. □

Neither space is sober, but each is missing just one point. This is easy to fix.

Definition 10.1.3. Let S be either of the spaces of Definition 10.1.1. Let

$${}^{+}S = S \cup \{\omega\}$$

where ω is a new point, the tag. In the same way let

$${}^{+}E = E \cup \{\omega\}$$

for each $E \subseteq S$. Let

$${}^{+}\mathcal{F}S = \{{}^{+}U \mid U \in \mathcal{F}S\}$$

to produce a filter on ${}^{+}S$, and let

$$\mathcal{O}^{+}S = \{\varnothing\} \cup {}^{+}\mathcal{F}S$$

to produce a topology on ${}^{+}S$. □

Almost trivially $\mathcal{O}^{+}S$ is a topology on ${}^{+}S$, and S is a subspace of ${}^{+}S$. In fact, we can say more.

Lemma 10.1.4. *For each space S of Definition 10.1.1, the insertion*

$$S \longrightarrow {}^{+}S$$

provides the sober reflection of S.

Proof. We need not go through all the details, but we should at least check that ${}^{+}S$ is sober. Since each non-empty open set of ${}^{+}S$ must contain ω, we see that the closure ω^{-} is the whole space.

A slight modification of the proof of Lemma 10.1.2 shows that each proper, closed irreducible subset of ${}^{+}S$ is a singleton. □

The topologies of a space and its sober reflection are canonically isomorphic. In these cases we see that

$$\begin{array}{rcll} \mathcal{O}S & \longrightarrow & \mathcal{O}^{+}S & \\ W & \longmapsto & {}^{+}W & \quad\text{for } W \in \mathcal{F}S \\ \varnothing & \longmapsto & \varnothing & \end{array}$$

is that isomorphism.

We also need the front topology of ${}^{+}S$.

Lemma 10.1.5. *For each space S of Definition 10.1.1 we have*

$$\mathcal{O}^{f+}S = \mathcal{P}S \cup {}^{+}\mathcal{F}S$$

that is, each front open subset of ${}^{+}S$ is either a subset of S or a tagged, open, non-empty subset of S.

Proof. The front topology has a base

$$U \cap X$$

for all open U and closed X. Since each $s \in S$ is a closed point of ${}^{+}S$ we see that

$$\mathcal{P}S \cup {}^{+}\mathcal{F}S \subseteq \mathcal{O}^{f+}S$$

and it suffices to show the converse inclusion.

Consider any basic open set $U \cap X$ of ${}^{f+}S$. If $U = \varnothing$ then $U \cap X = \varnothing \subseteq S$. Otherwise we have $U = {}^{+}F$ for some $F \in \mathcal{F}S$. If $X' \in \mathcal{F}S$ then $U \cap X \subseteq S$ (since the intersection is untagged). Otherwise $X = {}^{+}S$ and then $U \cap X = {}^{+}F \in \mathcal{F}S$. □

This sets down the basic properties of these two spaces. Next we want to describe the patch properties.

The point-sensitive patch properties

The major difference between the cofinite and the cocountable topologies, and the reason why we find the cocountable topology more useful for our purposes, lies in the compact saturated sets. Both the spaces S are T_1, so each subset is saturated. However, the compact sets are very different.

Lemma 10.1.6. *(a) Let S be an infinite set with the cofinite topology. Then $\mathcal{Q}S = \mathcal{P}S$.*

(b) Let S be an infinite set with the cocountable topology. Then $QS = \mathcal{P}_{\mathsf{fin}}S$, the collection of finite subsets.

Proof. (a) Let Q be any non-empty subset and let $\mathcal{U}$ be any open cover of Q. Since $Q \neq \varnothing$ there is at least one non-empty $U \in \mathcal{U}$. But now $Q - U$ is finite, and so can be covered by just finitely many members of $\mathcal{U}$.

(b) Consider any $Q \in \mathcal{Q}S$ and, by way of contradiction, suppose Q is infinite. Let X be any countably infinite subset of Q. Note that X' is open.

For each $y \in Q$ let

$$U_y = X' \cup \{y\}$$

to obtain an open set. Then

$$\mathcal{U} = \{U_y \mid y \in Q\}$$

covers Q and hence, by the compactness, we have

$$Q \subseteq U_{y_1} \cup \cdots \cup U_{y_n} = X' \cup \{y_1, \ldots, y_n\}$$

for some $y_1, \ldots, y_n \in Q$. This gives

$$X = Q \cap X \subseteq \{y_1, \ldots, y_n\}$$

which is the contradiction, since X is infinite. □

This result enables us to describe the patch topologies on both S and ${}^+S$. For this we introduce some notation.

Definition 10.1.7. For each of the two spaces S of Definition 10.1.1 let $\mathcal{G}S$ be the set of all subsets

$$U \cap (S - H)$$

for $U \in \mathcal{F}S$ and $H \in \mathcal{Q}S$. □

This $\mathcal{G}S$ is a filter on S. In fact, we see that

$$(a) \quad \mathcal{G}S = \mathcal{P}S \qquad (b) \quad \mathcal{G}S = \mathcal{P}_{\mathsf{cc}}S = \mathcal{F}S$$

for the two cases. Using this notation we have the following.

Theorem 10.1.8. *For each of the two spaces S of Definition 10.1.1, we have*

$$\mathcal{O}^p S = \{\varnothing\} \cup \mathcal{G}S \qquad \mathcal{O}^{p+} S = \{\varnothing\} \cup {}^+\mathcal{F}S \cup \mathcal{G}S$$

where $\mathcal{G}S$ is as in Definition 10.1.7.

Proof. The topology on ${}^{p}S$ is generated by the sets $U \cap Q'$ for $U \in \mathcal{O}S$ and $Q \in \mathcal{Q}S$. But this generating family is just

$$\{\varnothing\} \cup \mathcal{G}S$$

which is already a topology.

For the description of $\mathcal{O}^{p+}S$ we verify several inclusions.

The inclusions

$$\{\varnothing\} \cup {}^{+}\mathcal{F}S \subseteq \mathcal{O}^{+}S \subseteq \mathcal{O}^{p+}S$$

are immediate.

Consider any non-empty $Q \in \mathcal{Q}^{+}S$. The specialisation order of ${}^{+}S$ is the discrete set S with the tag ω sitting on top. Thus

$$Q = \{\omega\} \cup H$$

for some $H \subseteq S$. A simple argument shows that H is compact in S, and hence $S - H \in \mathcal{G}S$. Thus

$${}^{+}S - Q = S - H \in \mathcal{G}S$$

to show that

$$\{\varnothing\} \cup {}^{+}\mathcal{F}S \subseteq \mathcal{O}^{p+}S \subseteq \{\varnothing\} \cup {}^{+}\mathcal{F}S \cup \mathcal{G}S$$

holds. Since the larger family is a topology on ${}^{+}S$, it suffices to show that $\mathcal{G}S \subseteq \mathcal{O}^{p+}S$.

Consider any member of $\mathcal{G}S$. This has the form

$$G = U \cap (S - H) = U - H$$

where $U \in \mathcal{F}S$ and $H \in \mathcal{Q}S$. But now ${}^{+}H \in \mathcal{Q}^{+}S$, and hence

$$G = U - H =^{+} U - {}^{+}H \in \mathcal{O}^{p+}S$$

as required. □

It is worth restating these results separately for the two cases, for there is a nice surprise.

Theorem 10.1.9. *(a) Let S be an infinite set with the cofinite topology $\mathcal{O}S$. Then*

$$\mathcal{O}^{p}S = \mathcal{P}S \qquad \mathcal{O}^{p+}S = \mathcal{P}S \cup {}^{+}(\mathcal{P}_{\mathsf{cof}}S) = \mathcal{O}^{f}S$$

hold.

(b) Let S be an uncountable set with the cocountable topology. Then

$$\mathcal{O}^{p}S = \mathcal{O}S \qquad \mathcal{O}^{p+}S = \{\varnothing\} \cup {}^{+}\mathcal{F}S \cup \mathcal{F}S$$

and $\mathcal{O}^{p}S$ is just the cocountable topology on the tagged set ${}^{+}S$.

Proof. Only the description of $\mathcal{O}^{p+}S$ for case (b) is not immediate. But here we have

$$\mathcal{F}S = \mathcal{P}_{\mathsf{cc}}S \qquad \mathcal{G}S = \mathcal{F}S = \mathcal{P}_{\mathsf{cc}}S$$

so that ${}^{+}\mathcal{F}S \cup \mathcal{F}S$ is the family of cocountable subsets of ${}^{+}S$. □

Now for the surprise. Consider case (b), the uncountable set S with the cocountable topology. The patch space ${}^{p+}S$ is just the tagged space ${}^{+}S$ with the cocountable topology. But S and ${}^{p+}S$ have the same size and any bijection between them is a homomorphism. In particular, $\mathcal{O}S$ and $\mathcal{O}^{p+}S$ are isomorphic, but not canonically. This gives us the following result.

Theorem 10.1.10. *The process*

$$S \longmapsto {}^{p+}S$$

of taking the patch space of the sober reflection can continue forever.

This should be compared with the two processes

$$S \longmapsto {}^{+}S \qquad S \longmapsto {}^{p}S$$

of sobering up a space and taking its patch space. The first one stabilises after one step and, by Corollary 4.2.5, the second stabilises after two steps. However it seems that if we mix them then things can get nasty. I have no idea what happens at the first limit level.

The full assembly

In this subsection we continue the analysis of the two spaces of Definition 10.1.1. Thus S is a space of the appropriate cardinality and

$$\mathcal{O}S = \{\varnothing\} \cup \mathcal{F}S$$

is the cofinite or cocountable topology. Our aim is to describe the full assembly $N\mathcal{O}S$.

Definition 10.1.11. For each $H \subseteq S$ set

$$[H](U) = \begin{cases} H \cup U & \text{for } U \in \mathcal{F}S \\ H^{\circ} & \text{for } U = \varnothing \end{cases}$$

and

$$\langle H \rangle(U) = \begin{cases} H \cup U & \text{for } U \in \mathcal{F}S \\ \varnothing & \text{for } U = \varnothing \end{cases}$$

(for $U \in \mathcal{O}S$) to produce two functions from $\mathcal{O}S$ to itself. □

Notice that on general grounds $[H]$ is just the spatially induced nucleus on $\mathcal{O}S$ obtained from $H \subseteq S$. The special nature of S ensures that $\langle H \rangle$ is a nucleus also.

Lemma 10.1.12. *For all $H \subseteq S$ each of the operators $[H]$ and $\langle H \rangle$ is a nucleus on $\mathcal{O}S$.*

Proof. Only the $\wedge$-preserving property of $\langle H \rangle$ is not immediate. We need to check that

$$\langle H \rangle U \cap \langle H \rangle V = \langle H \rangle (U \cap V)$$

holds for each $U, V \in \mathcal{O}S$.

If either U or V is empty, then both sides are empty. If both U and V are non-empty then so is $U \cap V$ and hence

$$\langle H \rangle U \cap \langle H \rangle V = (H \cup U) \cap (H \cup V) = H \cup (U \cap V) = \langle H \rangle (U \cap V)$$

as required. □

As we let H vary through $\mathcal{P}S$ there will be some repetitions amongst the nuclei $[H]$ and $\langle H\rangle$. If $H = \varnothing$ or $H \notin \mathcal{F}S$ then $H^\circ = \varnothing$ and hence $[H] = \langle H\rangle$. We make a selection to avoid these duplications. Recall that from Lemma 10.1.5 we have

$$\mathcal{O}^{f+}S = \{\varnothing\} \cup (\mathcal{P}S - \{\varnothing\}) \cup {}^{+}\mathcal{F}S$$

where now it is convenient to separate $\varnothing$ from the rest. There are no repetitions amongst these sets, and we use them to index some selected nuclei.

Definition 10.1.13. The selected nuclei on $\mathcal{O}S$ are given by

$$\begin{array}{rcll} \mathcal{O}^{f+}S & \longrightarrow & N\mathcal{O}S & \\ {}^{+}W & \longmapsto & [W] & \text{for } W \in \mathcal{F}S \\ H & \longmapsto & \langle H\rangle & \text{for } H \in \mathcal{P}S - \{\varnothing\} \\ \varnothing & \longmapsto & \mathrm{id} & \end{array}$$

using the decomposition of $\mathcal{O}^{f+}S$. □

We will show that this injection is, in fact, a frame isomorphism, and then use this to connect the point-free and point-sensitive patch constructions.

First we check the morphism properties.

Lemma 10.1.14. *The equalities*

- $[W] \wedge [V] = [W \cap V] \qquad [W] \wedge \langle H\rangle = \langle W \cap H\rangle \qquad \langle K\rangle \wedge \langle H\rangle = \langle K \cap H\rangle$
- $\bigvee\{[W] \mid W \in \mathcal{W}\} = \dot{\bigvee}\{[W] \mid W \in \mathcal{W}\} = [\bigcup \mathcal{W}]$
- $\bigvee\{\langle H\rangle \mid H \in \mathcal{H}\} = \dot{\bigvee}\{\langle H\rangle \mid H \in \mathcal{H}\} = \langle\bigcup \mathcal{H}\rangle$

hold for all $V, W \in \mathcal{F}S$, all $K, H \in \mathcal{P}S - \{\varnothing\}$ and all $\mathcal{H} \subseteq \mathcal{P}S - \{\varnothing\}$.

Proof. This is just routine calculations. We will prove only the final part here as an illustration.

We show that

$$\dot{\bigvee}\{\langle H\rangle \mid H \in \mathcal{H}\} = \left\langle \bigcup \mathcal{H} \right\rangle$$

and then, since this is a nucleus, it must be the supremum we are looking for.

For each $U \in \mathcal{O}S$.

$$\begin{aligned} \bigcup\{\langle H\rangle U \mid H \in \mathcal{H}\} &= \begin{cases} \bigcup(H \cup U) & U \in \mathcal{F}S \\ \varnothing & U = \varnothing \end{cases} \\ &= \begin{cases} \bigcup\mathcal{H} \cup U & U \in \mathcal{F}S \\ \varnothing & U = \varnothing \end{cases} \\ &= \langle \bigcup \mathcal{H}\rangle \end{aligned}$$

which gives the result. □

By checking all the cases we see that this result shows that the assignment of Definition 10.1.13 is a frame embedding. This deals with the routine part of the following.

Theorem 10.1.15. *The assignment of Definition 10.1.13 is an isomorphism.*

Proof. Each nucleus in $N\mathcal{O}S$ is a supremum of nuclei of the form

$$[W] \wedge [V']$$

for $W, V \in \mathcal{O}S$ by Lemma 5.2.10. If $V = \varnothing$ then $V' = S$, and this infimum is $[W]$, therefore we may suppose $V \neq \varnothing$. But then either $V' = \varnothing$ or $V' \notin \mathcal{O}S$, so that $[V'] = \langle V' \rangle$ and hence

$$[W] \wedge [V'] = \langle W \cap V' \rangle$$

by Lemma 10.1.14. Hence each nucleus is of one of the forms

$$\bigvee \{[W] \mid W \in \mathcal{W}\} \qquad \bigvee \{\langle H \rangle \mid H \in \mathcal{H}\}$$

for some $\mathcal{W} \subseteq OS$ or some $\mathcal{H} \subseteq \mathcal{P}S$. Thus it is either $[W]$ or $\langle H \rangle$ where $W = \bigcup \mathcal{W}$ and $H = \bigcup \mathcal{H}$. We may exclude $W = \varnothing$ since $[\varnothing] = \langle \varnothing \rangle$.

To show uniqueness, we can just check the various cases. Suppose $[W] = [V]$. Then

$$W = [W]\varnothing = [V]\varnothing = V$$

as required. If $\langle H \rangle = \langle K \rangle$ for some $H, K \in \mathcal{P}S$ then consider any $x \in H$. The set $U = \{x\}'$ is open, and so

$$x \in H \cup \{x\}' = \langle H \rangle U = \langle K \rangle U = K \cup \{x\}'$$

so that $x \in K$. Hence $H \subseteq K$ and by symmetry, $K \subseteq H$. Finally, we check the case where $[W] = \langle H \rangle$ for some $W \in OS$ and $H \in \mathcal{P}S$. Then

$$W = [W]\varnothing = \langle H \rangle \varnothing = \varnothing$$

and we excluded this case earlier. □

Each frame carries three families

$$u. \qquad v. \qquad w.$$

of nuclei (which need not be disjoint).

For the topology $\mathcal{O}S$ these are easy to locate (although the notation is a bit tortuous).

Lemma 10.1.16. *On the frame* $\mathcal{O}S$

$$u_\varnothing = \mathrm{id} = \bot_{N\mathcal{O}S} \qquad v_\varnothing = \top_{N\mathcal{O}S} \qquad w_\varnothing = \langle S \rangle = \neg\neg$$

and

$$u_W = [W] \qquad v_W = \langle W' \rangle \qquad w_W = [W]$$

for each $\varnothing \neq W \in \mathcal{O}S$.

Proof. This is routine. We just prove two parts; the rest are similar.

We have

$$\begin{aligned} w_\varnothing(U) = (U \supset \varnothing) \supset \varnothing &= U'^{\circ} \supset \varnothing \\ &= \begin{cases} \varnothing \supset \varnothing & U \in \mathcal{F}S \\ S \supset \varnothing & U = \varnothing \end{cases} \\ &= \begin{cases} S & U \in \mathcal{F}S \\ \varnothing & U = \varnothing \end{cases} \\ &= \langle S \rangle (U) \end{aligned}$$

and we know that $(U \supset \varnothing) \supset \varnothing = \neg\neg(U)$ by definition, which gives the result.

For each $W \in \mathcal{F}S$,

$$v_W = [W'] = \langle W' \rangle$$

since $W'^{\circ} = \varnothing$. □

The patch assembly

In this subsection we obtain a complete description of the patch assembly $P\mathcal{O}S$ and match this with the patch topology $\mathcal{O}^{p+}S$. In particular, we show that $P\mathcal{O}S$ is spatial but, as yet, we do not locate all of its points.

Theorem 10.1.15 gives us a complete description of all the nuclei on $\mathcal{O}S$. We use this to locate the admissible filters. A few simple calculations gives the following.

Lemma 10.1.17. *We have*

$$\begin{aligned} U \in \nabla([W]) &\iff W' \subseteq U \\ U \in \nabla(\langle H \rangle) &\iff H' \subseteq U \text{ and } U \neq \varnothing \\ U \in \nabla(\mathrm{id}) &\iff U = S \end{aligned}$$

for each $W \in \mathcal{F}S$, $H \in \mathcal{P}S - \{\varnothing\}$ and $U \in \mathcal{O}S$.

Using this we can describe the blocks in $N\mathcal{O}S$.

Lemma 10.1.18. *For each $W \in \mathcal{F}S - \{S\}$ the two nuclei $\langle W \rangle$ and $[W]$ form a block with $\langle W \rangle \leq [W]$. All other blocks are singletons.*

Proof. It is easy to see that

$$\nabla([V]) = \nabla([W]) \iff V = W \qquad \nabla(\langle G \rangle) = \nabla(\langle H \rangle) \iff G = H$$

so we just need to check when $\nabla([W]) = \nabla(\langle H \rangle)$ which happens precisely when

$$(W' \subseteq U) \iff (H' \subseteq U \text{ and } U \neq \varnothing)$$

or in other words $W = H \neq S$. This proves the result. □

As a consequence of this we see that the fitted nuclei on $\mathcal{O}S$ are

$$\mathrm{id} \qquad \langle H \rangle \qquad [S]$$

for $H \subseteq S$. We have

$$\nabla(\mathrm{id}) = \{S\} \qquad \nabla([S]) = \mathcal{P}S$$

and the second of these is open but the first is not. The filter $\nabla(\langle H \rangle)$ may or may not be open depending on H.

To locate the open filters on $\mathcal{O}S$ we use the Hofmann-Mislove characterisation. Lemma 10.1.6 gives us $\mathcal{Q}S$, but this is not good enough; we need $\mathcal{Q}^+S$. A few calculations shows that $\mathcal{Q}^+S$ consists of

$$\{\omega\} \qquad {}^+Q \qquad \varnothing$$

for $Q \in \mathcal{Q}S$. Each of these produces an open filter on $\mathcal{O}S$, namely

$$\mathcal{F}S \qquad \nabla({}^{+}Q) \qquad \mathcal{P}S$$

respectively where

$$U \in \nabla({}^{+}Q) \Longleftrightarrow Q \subseteq U$$

for $U \in \mathcal{O}S$. These are the admitting filters of the nuclei

$$\langle S \rangle \qquad \langle Q' \rangle \qquad [S]$$

respectively. In other words, for $H \subseteq S$ the filter $\nabla(\langle H \rangle)$ is open if and only if $H = S$ or $H' \in \mathcal{Q}S$.

These calculations show that $P\mathcal{O}S$ is $\bigvee$-generated within $N\mathcal{O}S$ by the nuclei

$$\text{id} \qquad [W] \wedge \langle S \rangle = \langle W \rangle \qquad [W] \wedge \langle Q' \rangle = \langle W \cap Q' \rangle \qquad [W] \wedge [S] = [W]$$

for $W \in \mathcal{F}S$ and $Q \in \mathcal{Q}S$. This leads to the following companion of Theorem 10.1.15.

Theorem 10.1.19. *For each of the two spaces S of Definition 10.1.1, the assignment*

$$\begin{array}{rcll} \mathcal{O}^{p+}S & \longrightarrow & P\mathcal{O}S & \\ {}^{+}W & \longmapsto & [W] & \textit{for } W \in \mathcal{F}S \\ W & \longmapsto & \langle W \rangle & \textit{for } W \in \mathcal{G}S \\ \varnothing & \longmapsto & \text{id} & \end{array}$$

is a frame isomorphism.

As a consequence of this we see that for the space (a), the cofinite topology, we have

$$\mathcal{O}^{f+}S = \mathcal{O}^{p+}S \cong P\mathcal{O}S = N\mathcal{O}S$$

that is, the full and patch assemblies are the same. For the space (b), the cocountable topology, the patch assembly is strictly smaller than the full assembly. In both cases the isomorphism

$$\mathcal{O}^{p+}S \longrightarrow P\mathcal{O}S$$

shows that $P\mathcal{O}S$ is spatial and locates some of its points. However, since ${}^{p+}S$ is not sober, there are some missing points. What are these?

A wild point

In general there are two kinds of points of a patch assembly $P\mathcal{O}S$. Firstly there are the standard points w_p where p is a $\wedge$-irreducible of $\mathcal{O}S$. Secondly, there may be some wild points. For each such point l we have $u_p \leq l \leq w_p$ where $p = l(\perp)$. In particular, if $u_p = w_p$ then there is no wild point associated with p.

What are the standard points of $P\mathcal{O}S$? Each $\wedge$-irreducible of $\mathcal{O}S$ has the form $W = X'$ where X is a closed irreducible of S. Each such X is either a singleton $\{s\}$ or the whole space.

For $W = \{s\}'$ we have

$$u_W = w_W = [W]$$

and hence this has no associated wild point.

For $W = S' = \varnothing$ we have

$$w_W = \neg\neg = \langle S \rangle$$

with

$$u_W = \mathrm{id}$$

and, as we now show, this has an associated wild point.

Theorem 10.1.20. *Let S be an uncountable set with the cocountable topology. Apart from the standard points of $\mathcal{POS}$ (arising from the points of ${}^+S$) the only wild point is* id *which has $\langle S \rangle$ as it's associated standard point.*

Proof. The proof is just a series of simple calculations.

First observe that

$$\langle W \rangle \neq \mathrm{id} \qquad \langle W \rangle \leq \langle S \rangle \qquad [W] \not\leq \langle S \rangle$$

for each $W \in \mathcal{F}$.

To show that id is a point of $\mathcal{POS}$ remember that every other member of $\mathcal{POS}$ is a $[\cdot]$-nucleus or a $\langle\cdot\rangle$-nucleus. Can two of these meet to give id? Since for $U, V \in \mathcal{F}S$ we have

$$[V] \cap [U] = [V \cap U] \qquad \langle V \rangle \cap [U] = \langle V \cap U \rangle \qquad \langle V \rangle \cap \langle U \rangle = \langle V \cap U \rangle$$

we see that this cannot happen.

Finally, let l be any wild point of $\mathcal{POS}$. We have $\mathrm{id} \leq l \leq \langle S \rangle$, and $l \neq \langle S \rangle$ (since this is a standard point). Since $[W] \not\leq \langle S \rangle$ for each $W \in \mathcal{F}S$, we see that either $l = \mathrm{id}$ (which is what we want) or $l = \langle W \rangle$ for some $W \in \mathcal{F}S$. By way of contradiction, suppose $l = \langle W \rangle$. Consider distinct $s, t \in S$. We have

$$(W \cup \{s\}) \cap (W \cup \{t\}) = W$$

and both the left hand sets are in $\mathcal{F}S$, so that

$$\langle W \cup \{s\} \rangle \wedge \langle W \cup \{t\} \rangle = \langle W \rangle = l$$

and hence

$$\langle W \cup \{s\} \rangle \leq \langle W \rangle$$

which cannot be. ∎

At the moment this is essentially the only example we know of a wild point.

10.2 A subregular topology on the reals

This was the first example we found of a space that was patch trivial but not regular. We have since found many more examples of this, but we include this all the same because it is also an example of certain other properties. In particular it is a T_2 space whose topology is not fit.

Let S be the real numbers $\mathbb{R}$ furnished with the topology $\mathcal{O}S$ consisting of all sets

$$U \cup (\mathbb{Q} \cap V)$$

where U and V are metric open sets. We may insist that U is a subset of V.

Note that in this topology $\mathbb{Q}$ is an open set.

Lemma 10.2.1. *The space S is T_2.*

Proof. The topology $\mathcal{O}S$ contains a subtopology (the metric topology) which is T_2 and is therefore itself T_2. □

Lemma 10.2.2. *The topology $\mathcal{O}S$ is not fit.*

Proof. Suppose for a contradiction that $\mathcal{O}S$ is fit. Then by the definition of fitness, since $\mathbb{Q} \not\subseteq \varnothing$ we can find metric open sets U, V and W such that

1. $\mathbb{Q} \cup U = \mathbb{R}$
2. $V \cup (W \cap \mathbb{Q}) \neq \varnothing$
3. $U \cap (V \cup (W \cap \mathbb{Q})) = \varnothing$

hold. From (3) we deduce that

$$U \cap V = U \cap W \cap \mathbb{Q} = \varnothing$$

and so $U \cap W$ is empty, since every non empty metric open contains a rational point. But then (1) shows we must have $V, W \subseteq \mathbb{Q}$ and so $W = V = \varnothing$. This contradicts (2), and therefore $\mathcal{O}S$ is not fit. □

10.3 The maximal compact topology

This is example 99 from [17]. It was the first example we found of a space that was tidy but not 1-tidy, and inspired the cocountable tree examples in Chapter 11.

Let

$$S = \{x, y\} \cup \mathbb{N}^2$$

where x, y are distinct points not in $\mathbb{N}^2$. Define

$$R_n = \{(m, n) \mid m \in \mathbb{N}\}$$

so that R_n is the nth row of $\mathbb{N}^2$. Then we let each point $p \in \mathbb{N}^2$ be open (so that the restriction of the topology to $\mathbb{N}^2$ is discrete). Then we take the open neighbourhoods of x and y as follows.

• Given $x \in U \subseteq S$, let U be open precisely when $U \cap R_n$ is cofinite for each $n \in \mathbb{N}$.

• Given $y \in V \subseteq S$, let V be open precisely when $R_n \subseteq V$ for all except finitely many $n \in \mathbb{N}$.

It is easy to check that this gives a topology.

Lemma 10.3.1. *The maximal compact topology is T_1 but not T_2.*

Proof. The space is T_1 since every singleton is closed. Only the pair $\{x, y\}$ doesn't have a T_2 separation. For suppose that $x \in U \in \mathcal{O}S$ and $y \in V \in \mathcal{O}S$. Then there is some n (in fact, cofinitely many) with $R_n \subseteq V$. For this n the set $U \cap R_n$ is cofinite, hence $U \cap V \neq \varnothing$. □

Lemma 10.3.2. *The maximal compact topology is sober.*

Proof. Consider any closed irreducible set P. We show that P is a singleton. Suppose that $p \in P \cap \mathbb{N}^2$. Then, since $\{q\}$ is open for each $q \in \mathbb{N}^2$, we have $P \cap \mathbb{N}^2 = \{p\}$. The set

$$U = \{x, y\} \cup \{p\}'$$

is open and $U \cap \{p\} = \varnothing$ so we have $P \cap U = \varnothing$ and hence $P = \{p\}$. Now suppose that $P = \{x, y\}$. The two sets

$$U = \{x\} \cup \mathbb{N}^2 \qquad V = \{y\} \cup \mathbb{N}^2$$

are open with

$$P \cap U \cap V = \varnothing$$

and hence one of

$$P \cap U = \varnothing \qquad P \cap V = \varnothing$$

holds. Hence $P \neq \{x, y\}$ as required. □

Lemma 10.3.3. *The maximal compact topology is compact.*

Proof. If $x, y \in W \in \mathcal{O}S$ then W' is finite. Because $y \in W$, the points of W' occur in just finitely many rows R_n. Because $x \in W$, the set $R_n - W$ is finite for each such n. This shows that S is compact. □

Lemma 10.3.4. *The maximal compact topology (viewed as a frame of open sets) is fit.*

Proof. Consider open sets $A \nsubseteq B$. We must produce open sets U, V such that

$$A \cup U = S \qquad U \cap V \subseteq B \qquad V \nsubseteq B$$

hold. Suppose first that $A \cap \mathbb{N}^2 \nsubseteq B$, and consider any $p \in \mathbb{N}^2$ with $p \in A - B$. Let $U = \{p\}'$ and $V = \{p\}$ to get

$$A \cup U = S \qquad U \cap V = \varnothing \qquad V \nsubseteq B$$

as required. Thus we may suppose $A \cap \mathbb{N}^2 \subseteq B$ and hence one of x, y is in $A - B$. With the other $z \in \{x, y\}$, let

$$U = \{z\} \cup \mathbb{N}^2 \qquad V = A - \{z\}$$

to produce open sets. Since U is missing just one point, which is in A, we have $A \cup U = S$. Since V contains a point of A which is not in B, we have $V \nsubseteq B$. Finally

$$U \cap V = A \cap \mathbb{N}^2 \subseteq B$$

as required. □

Lemma 10.3.5. *The maximal compact topology is packed. In fact, the compact sets are exactly the closed sets.*

Proof. Suppose there is a compact set E with $E^- \neq E$. Each neighbourhood of each member of $E^- - E$ must meet E, and so $E^- - E \subseteq \{x, y\}$ as the topology on $\mathbb{N}^2$ is discrete. Suppose that $y \in E^- - E$. Then E must contain points from infinitely many rows. Pick one point from each row; this collection of singletons together with the remainder of S gives an open covering of E with no finite subcover. Similarly, if $x \in E^- - E$, then E must contain an infinite number of points from some one row, so cover E by singletons from this row and the remainder of S. Once again, this has no finite subcover. Hence if $E^- \neq E$ then E is not compact.

The whole space is compact, therefore every closed subset is compact. □

The closed (compact) sets of S are

- finite subsets of $\mathbb{N}^2$
- {x} with a subset of finitely many rows of $\mathbb{N}^2$
- {y} with at most finitely many points from each row
- {x,y} with any subset of $\mathbb{N}^2$.

Lemma 10.3.6. *The maximal compact topology is 1-regular.*

Proof. Open filters are in bijective correspondence with compact saturated sets. For each $Q \in \mathcal{Q}S$ we check that

$$f\bot = \dot{\bigvee}\{v_U(\bot) \mid Q \subseteq U\} = d(1)$$

holds as follows

- $x, y \notin Q \Longrightarrow f\bot = Q'$
- $x \in Q, y \notin Q \Longrightarrow f\bot = Q' - \{y\}$
- $y \in Q\ x \notin Q \Longrightarrow f\bot = Q' - \{x\}$
- $x, y \in Q \Longrightarrow f\bot = Q'$

for the four cases, respectively.

Now, we need to show that for every pair $U \nsubseteq V \in \mathcal{O}S$ and $Q \in \mathcal{Q}S$ with $Q \subseteq U$, there are $X, Y \in \mathcal{O}S$ such that

$$X \cap Y \subseteq d(1) \qquad U \cup X = \top \qquad Y \subseteq U \qquad Y \nsubseteq V$$

hold.

• If $x, y \in U$ then set $X = S - U$, $Y = U$. Then $X \cap Y = \varnothing$ and the other requirements hold.

• If $x \in U, y \notin U$ then set $X = Q'$, $Y = U$, so that

$$U \cup Q' = S \qquad U \cap Q' \subseteq Q' - \{y\}$$

as required.

• If $y \in U, x \notin U$ then set $X = Q'$, $Y = U$ as above.

• If $x, y \notin U$ then there exists $p \in U \cap \mathbb{N}^2$ with $p \notin V$. Set $Y = \{p\}$, $X = S - \{p\}$. Then $X \cap Y = \varnothing$. □

This shows that the maximal compact topology is 2-tidy, yet we saw in Lemma 10.3.1 that it is not T_2 and therefore not 1-tidy.

10.4 A glueing construction

Let A be a frame. The set $A \times A$ of ordered elements taken from A is a frame in the obvious way. We use a pre-nucleus on A to extract a subframe.

Let

$$d\colon A \longrightarrow A$$

be such a pre-nucleus. We take the set of all pairs

$$(x, y)$$

from A where both

$$x \leq d(y) \qquad y \leq d(x)$$

hold. It is easy to check this gives a subframe of $A \times A$.

This is a simple example of the glueing construction.

Let S be any topological space. Let

$$S_+ = S_0 + S_1$$

be a disjoint sum of two copies of S. We furnish S_+ with a topology.

Let d be any pre-nucleus on $\mathcal{O}S$ for which

$$d(\{x\}') = S$$

for each $x \in S$. We will see why this extra condition is needed shortly.

Definition 10.4.1. Let X be a subset of a topological space S.

A point $x \in X$ is *isolated* in X if there is some $U \in \mathcal{O}S$ with $X \cap U = \{x\}$. Let $\mathsf{iso}(X)$ be the set of isolated points of X.

A point $x \in X$ is a *limit point* of X if it is not isolated in X. □

Example 10.4.2. For each $X \in \mathcal{C}S$ let

$$\mathsf{lim}(X)$$

be the set of limit points of X, the non-isolated points of X. We check that

- $\mathsf{lim}(X) \in \mathcal{C}S$
- $\mathsf{lim}(X) \subseteq X$
- $X \subseteq Y \Longrightarrow \mathsf{lim}(X) \subseteq \mathsf{lim}(Y)$
- $\mathsf{lim}(X \cup Y) = \mathsf{lim}(X) \cup \mathsf{lim}(Y)$

and hence

$$\mathsf{der}(U) = \mathsf{lim}(U')'$$

gives a pre-nucleus der on OS. Furthermore

$$\mathsf{der}(\{x\}') = \mathsf{lim}(x)' = \varnothing' = S$$

so that der is an example of the kind of pre-nucleus we need. □

The analogue of der can be set up on any frame A. For $a \in A$

$$b = \mathsf{der}(a)$$

is the largest element such that the interval $[a, b]$ is boolean.

The closure ordinal of der is the CB-rank of S. This can be arbitrarily large.

We use the special kind of d to furnish

$$S_+ = S_0 + S_1$$

with a topology. We take $\mathcal{O}S_+$ to be the set of all disjoint sums

$$U_0 + U_1$$

where

$$U_i \in \mathcal{O}S_i \qquad U_i \subseteq d(U_{1-i})$$

for $i = 0, 1$. This is just the concrete case of the glueing construction and, as there, gives a topology.

Lemma 10.4.3. *If S is T_2 then S_+ is T_1 and sober.*

Proof. Consider distinct points in S_+. If both lie in the same component then that component gives a T_2 separation. Suppose $x_0 \in S_0$ and $y_1 \in S_1$ are such that when viewed as points $x, y \in S$ we have $x \neq y$, so that

$$x \in U \qquad y \in V \qquad U \cap V = \varnothing$$

for some $U, V \in \mathcal{O}S$. The open sets

$$U_0 + U_1 \qquad V_0 + V_1$$

(where $U_0 = U_1 = U$ and $V_0 = V_1 = V$) give a T_2 separation of x_0 and y_1.

Now suppose $x_0 \in S_0$ and $x_1 \in S_1$ arise from the same point $x \in S$. The two sets

$$S_0 + \{x_1\}' \qquad \{x_0\}' + S_1$$

are open (by the special property of d) and these provide a T_1 (but not T_2) separation.

Consider any closed irreducible subset of S_+. This has the form

$$Z = Z_0 + Z_1$$

where Z_i is closed in S_i and Z_i is either empty or irreducible in S_i. Of course, at least one of Z_0, Z_1 is non-empty.

If Z_i is non-empty, then it is a singleton (since S is T_2). Thus the largest $Z_0 + Z_1$ can be is

$$Z = \{x_0, y_1\}$$

where $x_0 \in Z_0$ and $y_1 \in Z_1$. Both the open sets

$$S_0 + \{y_1\}' \qquad \{x_0\}' + S_1$$

meet Z, but the intersection

$$\{x_0\}' + \{y_1\}'$$

doesn't. Thus Z must be a singleton. $\square$

Each of the two components of S_+ gives a nucleus $[S_i]$ on $\mathcal{O}S_+$.

$$[S_0](U_0 + U_1) = (S_0 + U_1)^\circ = d(U_1) + U_1$$

$$[S_1](U_0 + U_1) = (U_0 + S_1)^\circ = U_0 + d(U_0)$$

We look at the join

$$[S_0] \vee [S_1]$$

of these nuclei.

Let

$$f = [S_0]\dot{\vee}[S_1] \qquad g = [S_0] \circ [S_1] \qquad h = [S_1] \circ [S_0]$$

that is

$$\begin{aligned} f(U_0 + U_1) &= d(U_1) + d(U_0) \\ g(U_0 + U_1) &= d^2(U_0) + d(U_0) \\ h(U_0 + U_1) &= d(U_1) + d^2(U_1) \end{aligned}$$

for the relevant U_0, U_1. On general grounds we have

$$f \le g, h \le f^2$$

and f^∞ is the required join.

In this case we can also consider

$$k(U_0 + U_1) = d(U_0) + d(U_1)$$

so that

$$k \le g, h \qquad k^2 = f^2$$

hold.

The closure ordinal of each of f, g, h, k is the closure ordinal of d and this can be arbitrarily large.

Observe that

$$[S_0] \vee [S_1] = \top \iff S \text{ is scattered}$$

holds.

This example was specifically constructed as a strongly stacked space for which

$$P\mathcal{O}S \xrightarrow{\pi} \mathcal{O}^p S$$

is not an isomorphism. With the usual notation

$$v_F = [Q']$$

for each open filter F since S is strongly stacked, or in other words π is an isomorphism when restricted to the basic elements of $P\mathcal{O}S$. However, things go wrong when we look at joins of basic nuclei, and we have demonstrated that there are patch nuclei j, k and l for which

$$\pi(j \vee k) = \pi(l)$$

and yet

$$j \vee k \neq l$$

holds. Equivalently, there are nuclei $j, k \in P\mathcal{O}S$ such that

$$\pi(j) \leq \pi(k)$$

yet $j \not\leq k$.

This has a number of implications for the patch assembly and its point space. It is a T_1 sober space, therefore it has no wild points and the point space of $P\mathcal{O}S$ is just ${}^{p}S$. We have shown that π is not an isomorphism, therefore $P\mathcal{O}S$ is not spatial.

The moral is that we cannot just look at the action of π on the basic nuclei because even when this is nice, there is still room for some bad behaviour.

Chapter 11

The boss topology on a tree

We first constructed these examples to answer the question of whether a space is packed exactly when it has a tidy topology. We know that

$$\mathcal{O}S \text{ tidy} \Longrightarrow S \text{ packed}$$

in general. We want to know whether the converse implication holds.

We will construct a variety of different tree topologies, but they all have a number of features in common. We will show that each (pointed) tree carries a

$$T_1 + \text{ sober } + \text{ tightly packed}$$

topology.

We also use these trees to give examples of spaces with different *tidiness* (see Section 8.2).

11.1 Trees and boss topologies

Trees are well known but we should start by reviewing a few standard notions.

Definition 11.1.1. A *tree* is a poset $\mathbb{S}$ such that for each *node* $x \in S$ the set

$$P(x) = \{y \in \mathbb{S} \mid y \leq x\}$$

of *predecessors* of x is linearly ordered. □

Sometimes a tree is required to have a stronger property, namely that each set $P(x)$ is well-ordered. In those circumstances what we here call a tree is a tree-like poset. As it happens the general machinery does not require the well-foundedness. However, all but one of the particular examples will be well founded. For most of our examples $P(x)$ will be finite, however we will later see that this need not be the case.

For the time being we work with an arbitrary tree $\mathbb{S}$. As usual, we write $<$ for the strict comparison on $\mathbb{S}$, that is

$$x < y \Longleftrightarrow x \leq y \text{ and } x \neq y$$

for $x, y \in \mathbb{S}$.

Definition 11.1.2. For $x, y \in \mathbb{S}$ we say x is an *immediate predecessor* of y or y is an *immediate successor* of x if $x < y$ and

$$x \le z \le y \Longrightarrow x = z \text{ or } z = y$$

for each $z \in \mathbb{S}$. In other words, there is a gap between x and y. Let

$$I(x) = \{y \in \mathbb{S} \mid y \text{ is an immediate successor of } x\}$$

for each $x \in \mathbb{S}$. □

Notice that the tree and gap property give

$$I(x_1) \cap I(x_2) \neq \varnothing \Longrightarrow x_1 = x_2$$

for each $x_1, x_2 \in \mathbb{S}$.

The technique we develop uses the size of subsets of $I(x)$ for arbitrary nodes $x \in \mathbb{S}$. For this it is convenient to have a convention.

For the purposes of these examples, we say that for each $x \in \mathbb{S}$ (the parent tree) a subset $E \subseteq I(x)$ is *small* if it is countable and *large* if it is uncountable.

There are many other possible interpretations of 'small' and 'large' that could be used. The crucial properties a notion of smallness must have are the following.

- Each singleton and $\varnothing$ is small.
- $H \subseteq K$ with K small $\Longrightarrow$ H is small.
- H, K small $\Longrightarrow$ $H \cup K$ is small.

Then a set is 'large' exactly when it is not small. Not all these notions will give us the properties these examples have that we are interested in, however. In particular an interpretation of 'small' as 'finite' will not work. We will locate the crucial step that fails for this interpretation.

To analyse the tree $\mathbb{S}$ we attach a *boss point* to form

$$S = \mathbb{S} \cup \{*\}$$

(where $* \notin \mathbb{S}$), and then we impose a topology on S.

Definition 11.1.3. Let $\mathcal{O}S$ be the family of subsets $U \subseteq S$ such that both

$$(\forall x \in \mathbb{S})[x \in U \Longrightarrow I(x) - U \text{ is small}]$$

$$* \in U \Longrightarrow (\forall x \in \mathbb{S})[I(x) - U \text{ is small}]$$

hold. □

It is easy to show that this is a topology.

Lemma 11.1.4. *The family $\mathcal{O}S$ is a topology on S.*

Note that a subset $X \subseteq S$ is closed precisely when both

$$(\forall x \in \mathbb{S})[x \notin X \Longrightarrow I(x) \cap X \text{ is small}]$$

$$* \notin X \Longrightarrow (\forall x \in \mathbb{S})[I(x) \cap X \text{ is small}]$$

hold or equivalently when both

$$(\exists x \in \mathbb{S})[I(x) \cap X \text{ is large}] \Longrightarrow * \in X$$

$$(\forall x \in \mathbb{S})[I(x) \cap X \text{ is large}] \Longrightarrow x \in X$$

hold.

Definition 11.1.5. For each tree $\mathbb{S}$ we call the associated space S the *boss space* of $\mathbb{S}$ with $\mathcal{O}S$ the *boss topology*. The extra point $*$ is the *boss* of the space. □

We will see how the boss $*$ controls many of the properties of $\mathbb{S}$ and so explain the terminology.

We will need some examples of open sets and closed sets.

Example 11.1.6. (a) For each $y \in \mathbb{S}$ the principal upper section

$$U = \uparrow y = \{z \in \mathbb{S} \mid y \leq z\}$$

is open. By construction $* \notin U$ so it suffices to consider those $x \in U$. For such an x we have $I(x) \subseteq U$ so that $I(x) - U = \varnothing$ which is certainly small.

(b) For each open set V with $* \notin V$ and each $y \in \mathbb{S}$ the set

$$U = V - \uparrow y$$

is open. Since $* \notin U$ it suffices to consider those $x \in U$. For such an x we have

$$I(x) - U = (I(x) - V) \cup (I(x) \cap \uparrow y)$$

and the first component is small (since $x \in V \in \mathcal{O}S$). Consider any z in the second component. Thus z is an immediate successor of x and $y \leq z$, hence $x < y \leq z$ (since $\mathbb{S}$ is a tree and $y \nleq x$). The gap property gives $z = y$. Thus the second component is no more than a singleton.

(c) Consider a countable subset $X \subseteq \mathbb{S}$. For each $x \in \mathbb{S}$ the set $I(x) \cap X$ is small, and hence X is closed.

(d) Consider any countable $Z \subseteq \mathbb{S}$ and any $Y \subseteq \downarrow Z$. In other words, for each $y \in Y$ there is some $z \in Z$ with $y \leq z$. Such a Y is closed. To see this consider any $y \in I(x) \cap Y$ (where $x \in \mathbb{S}$). Then x is an immediate predecessor of y and $y \leq z$ for some $z \in Z$. For this z suppose x has immediate successors $y_1, y_2 \in Y$ such that $y_1, y_2 \leq z$. By the tree property we may suppose $x < y_1 \leq y_2 \leq z$, but then $y_1 = y_2$ since x is an immediate predecessor of y_2. This shows there is an injection

$$I(x) \cap Y \longrightarrow Z$$

so that $I(x) \cap Y$ is countable, and hence small. □

Using these examples we can obtain the fundamental property of S.

Lemma 11.1.7. *The space S is T_1 and sober.*

Proof. For $x \in \mathbb{S}$ the set $\{x\}$ is countable and closed (by Example 11.1.6(c)). Also

$$\{*\}' = \mathbb{S} = \bigcup \{\uparrow y \mid y \in \mathbb{S}\}$$

which is open. This shows that every point of S is closed, and hence S is T_1.

To see that S is sober consider any closed irreducible set Z. We must show that Z is a singleton. To do that we first show that $Z \cap \mathbb{S}$ is no more than a singleton.

By way of contradiction suppose there are distinct $x, y \in Z \cap \mathbb{S}$. Set $U = \uparrow x$ and $V = \uparrow y$. Both U and V are open, and since

$$x \in Z \cap U \qquad y \in Z \cap V$$

the irreducibility of Z ensures that

$$Z \text{ meets } U \cap V$$

to give $x, y \leq z \in Z$ for some $z \in \mathbb{S}$. Since $\mathbb{S}$ is a tree we may suppose $x < y \leq z$ (by symmetry). The set

$$W = \uparrow x - \uparrow y$$

is open (by Example 11.1.6(a, b)) and

$$x \in W \cap Z$$

so that

$$Z \text{ meets } W \cap V$$

(by a second use of the irreducibility). Since

$$W \cap V \subseteq V' \cap V = \varnothing$$

this gives the required contradiction.

This shows that either Z is a singleton or has the form $\{*, x\}$ for some $x \in \mathbb{S}$. We must exclude this second case.

Suppose $Z = \{*, x\}$. Then

$$x \in Z \cap \uparrow x \qquad * \in Z \cap \{x\}'$$

with both $\uparrow x$ and $\{x\}'$ open. Thus

$$Z \text{ meets } \uparrow x \cap \{x\}'$$

by a third use of irreducibility. This gives some $z \in Z \cap \mathbb{S}$ with $x < z$, which can not be since $Z \cap \mathbb{S}$ is no more than a singleton. □

Our next task is to describe the family $\mathcal{Q}S$ of compact saturated subsets of S. Since S is T_1 these are the compact subsets of S. As usual, each finite subset is compact, but here we have the converse.

The proof of the following is the crucial step where we need

$$\text{countable} \implies \text{small}$$

as the proof will not work if 'small' means 'finite'.

Lemma 11.1.8. *For the space S each compact subset is finite.*

Proof. Let $Q \in \mathcal{Q}S$. We prove that Q is finite in two steps. First we show that certain sections of Q are finite, then we go for the whole set.

Recall that a subset $L \subseteq \mathbb{S}$ is an antichain if

$$y \leq z \implies x = y$$

holds for all $y, z \in L$. For example, for each $x \in \mathbb{S}$ the set $I(x)$ is an antichain.

We show first that for each antichain L of $\mathbb{S}$, the intersection $Q \cap L$ is finite.

By way of contradiction suppose $Q \cap L$ is infinite. Thus there is a countably infinite subset $X \subseteq Q \cap L$. This set X is closed (by Example 11.1.6(c)). Thus

$$X' \cup \{\uparrow x \mid x \in X\}$$

is an open cover of S (since $* \in X'$), and hence of Q. The compactness gives

$$X \subseteq Q \subseteq X' \cup \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

for some $x_1, \ldots, x_m \in X$ and hence

$$X \subseteq L \cap (\uparrow x_1 \cup \cdots \cup \uparrow x_m)$$

holds. Consider any $y \in X$. Then $y \in L$ and $x_i \le y$ for some $1 \le i \le m$. But $x_i \in X \subseteq L$ and L is an antichain, so that $y = x_i$. This gives

$$X \subseteq \{x_1, \ldots, x_m\}$$

which is the contradiction (since X was supposed to be infinite).

Secondly we use this observation to show that Q is finite.

Consider any subset $H \subseteq S$. For each $x \in \mathbb{S}$ the set $I(x)$ is an antichain and so $I(x) \cap Q \cap H$ is finite and hence small. This shows that $Q \cap H$ is closed. In particular for each $q \in Q$ the set

$$X_q = Q \cap \{q\}'$$

is closed, and hence

$$U_q = X_q{}' = Q' \cup \{q\}$$

is open. But

$$\{U_q \mid q \in Q\}$$

is an open cover of S, so a second use of the compactness of Q gives

$$Q \subseteq Q' \cup \{q_1, \ldots, q_m\}$$

for some $q_1, \ldots q_m \in Q$. Thus

$$Q \subseteq \{q_1, \ldots, q_m\}$$

to give the required result. □

Lemmas 11.1.7 and 11.1.8 combine to give the following.

Theorem 11.1.9. *For each tree $\mathbb{S}$ the associated boss space S is T_1, sober, and tightly packed.*

This result suggests that the boss space of a tree is quite nice. However, we will show that such a space can be quite a way from being T_2. We do that by analysing the stacking properties.

11.2 Stacking properties of a boss space

We will show that for suitably chosen $\mathbb{S}$ the space S can be stacked with an arbitrarily large degree of tidiness. To do that we need some properties of the derivatives $\widehat{Q}$ for $Q \in \mathcal{Q}S$. These operators were first introduced in Section 5.6. Our eventual aim is to show how each of these is controlled by the particular case $\widehat{\{*\}}$ obtained from $Q = \{*\}$.

Lemma 11.2.1. *For each $Q \in \mathcal{Q}S$ we have*

$$\widehat{Q}(Q) = Q \qquad \widehat{Q}(Q \cup X) = Q \cup \widehat{Q}(X)$$

for each $X \in \mathcal{C}S$.

Proof. We have

$$\widehat{Q}(Q) = \bigcap\{(Q \cap U)^- \mid Q \subseteq U \in \mathcal{O}S\} = Q^- = Q$$

since Q is closed. With this

$$\widehat{Q}(Q \cup X) = \widehat{Q}(Q) \cup \widehat{Q}(X) = Q \cup \widehat{Q}(X)$$

as required. □

We need a fuller description of the behaviour of $\widehat{Q}$ on certain closed sets.

Definition 11.2.2. For each $X \subseteq S$ we use

$$x \in \odot(X) \Longleftrightarrow x \in \mathbb{S} \cap X \text{ and } I(x) \cap X \text{ is small}$$

(for $x \in \mathbb{S}$) to extract a subset $\odot(X)$ of X.

We set

$$X^* = X - \odot(X)$$

to produce the complement subset. □

By definition of the topology we have

$$* \notin X \in \mathcal{C}S \Longrightarrow \odot(X) = X \Longrightarrow X^* = \varnothing$$

but if $* \in X \in CS$ then X^* may be non-empty. However, we do have the following.

Lemma 11.2.3. *For each $X \in \mathcal{C}S$ the set X^* is closed.*

Proof. If $* \notin X$ then $X^* = \varnothing$ which is closed.

Suppose $* \in X$, so that $* \in X^*$ (since $* \notin \odot(X)$). We have

$$I(y) \cap X^* \subseteq I(y) \cap X$$

so it suffices to show that the larger set is small for each $y \in \mathbb{S} - X^*$. Consider such a y. If $y \notin X$ then $I(y) \cap X$ is small (since X is closed). If $y \in X - X^*$ then $y \in \odot(X)$, and hence $I(y) \cap X$ is small (by definition of $\odot(X)$). □

The following indicates why the set $\odot(X)$ is useful and why the point $*$ is so bossy.

Lemma 11.2.4. *For each $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$ we have*

$$X \cap Q \subseteq \widehat{Q}(X) \subseteq X^* \cup (X \cap Q)$$

with

$$\widehat{Q}(X) = X^* \cup (X \cap Q)$$

when $* \in Q$.

Proof. The left hand inclusion is immediate.

For the right hand inclusion we have

$$X = X^* \cup \odot(X) = X^* \cup (Q \cap \odot(X)) \cup (\odot(X) - Q) \subseteq X^* \cup (X \cap Q) \cup (\odot(X) - Q)$$

so that an implication

$$z \in \odot(X) - Q \Longrightarrow z \notin \widehat{Q}(X)$$

will suffice. To this end consider any $z \in \odot(X) - Q$ and observe that

$$z \in X \qquad I(z) \cap X \text{ is small} \qquad z \notin Q$$

hold. Let $U = \{z\}'$ so that U is open with $Q \subseteq U$ and $* \in U$. We show that $X \cap U$ is closed, so that

$$\widehat{Q}(X) \subseteq (X \cap U)^- = X \cap U = X - \{z\}$$

and hence $z \notin \widehat{Q}(X)$. Since $(X \cap U)^- \subseteq X$ it suffices to show that

$$I(x) \cap X \text{ is small}$$

for all $x \notin (X \cap U)$.

Firstly, for each $x \in \mathbb{S}$ we have

$$* \notin X \cap U \Longrightarrow * \notin X \Longrightarrow I(x) \cap X \text{ is small}$$

since X is closed.

Secondly, consider $x \in \mathbb{S} - (X \cap U)$. If $x \notin X$ then $I(x) \cap X$ is small since X is closed. If $x \notin U = \{z\}'$ then $x = z$ and so $x \in \odot(X)$ (by choice of z) which again leads to the required result.

To prove the equality it suffices to show that $X^* \subseteq \widehat{Q}(X)$ whenever $* \in Q$.

If $* \notin X$ then $X^* = \varnothing$ and then this inclusion certainly holds. Thus we may suppose $* \in X$.

Consider any $U \in \mathcal{O}S$ with $Q \subseteq U$ and let $V = (X \cap U)^{-\prime}$. We show that

$$X \cap V \subseteq \odot(X)$$

so that

$$X \subseteq (X \cap U)^- \cup \odot(X)$$

and hence

$$X^* = X - \odot(X) \subseteq (X \cap U)^-$$

which, since U is arbitrary, gives the result.

Observe that

$$X \cap U \cap V \subseteq V' \cap V = \varnothing$$

so that $X \subseteq U' \cup V'$ and hence

$$I(x) \cap X \subseteq I(x) \cap (U' \cup V') = (I(x) \cap U') \cup (I(x) \cap V')$$

for each $x \in \mathbb{S}$. Consider any $x \in X \cap V$. Then both

$$I(x) \cap U' \qquad I(x) \cap V'$$

are small (since $* \in Q \subseteq U$ in the left hand case and since V is open in the right hand case). Thus $I(x) \cap X$ is small, and hence $x \in \odot(X)$ (since $x \in X$), as required to complete the proof. □

One particular example of $Q \in \mathcal{Q}S$ is the singleton $\{*\}$. This explains our use of the notation X^*.

Corollary 11.2.5. *For each $X \in \mathcal{C}S$ we have $\widehat{\{*\}}(X) = X^*$.*

Proof. We have

$$\widehat{Q}(X) = X^* \cup (X \cap \{*\})$$

by Lemma 11.2.4. If $* \notin X$ then $X \cap \{*\} = \varnothing$. If $* \in X$ then

$$X \cap \{*\} = \{*\} \subseteq X^*$$

(since $* \notin \odot(X)$). In either case the extra right hand component is absorbed by the left hand component. □

Each $Q \in \mathcal{Q}S$ gives a descending chain

$$\mathbf{Q} = \{Q(\alpha) \mid \alpha \in \mathsf{Ord}\}$$

of closed sets each of which includes Q. Thus, as usual,

$$Q(\alpha) = \widehat{Q}^{\alpha}(S)$$

for each ordinal α. On cardinality grounds there is some ordinal ∞ such that $Q(\alpha) = Q(\infty)$ for all ordinals $\alpha \geq \infty$ and we then have $Q \subseteq Q(\infty)$. We are interested in the size of ∞ and whether or not $Q(\infty) = Q$. Lemma 11.2.4 indicates that we should pay special attention to the boss case $Q = \{*\}$.

Definition 11.2.6. Let

$$S^{(\alpha)} = \widehat{\{*\}}^{\alpha} S$$

for each ordinal α. Thus

$$S^{(0)} = S \qquad S^{(\alpha+1)} = S^{(\alpha)*} \qquad S^{(\lambda)} = \bigcap\{(S^{(\alpha)} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. □

This particular chain provides an upper bound for every Q-chain.

Lemma 11.2.7. *For each $Q \in \mathcal{Q}S$ we have*

$$Q(\alpha) \subseteq S^{(\alpha)} \cup Q$$

with equality if $* \in Q$.

Proof. We proceed by induction on α. For the base case, $\alpha = 0$, we have

$$Q(0) = S = S^{(0)} \cup Q$$

as required.

For the induction step, $\alpha \mapsto \alpha + 1$, the induction hypothesis gives

$$Q(\alpha + 1) \subseteq \widehat{Q}(S^{(\alpha)} \cup Q) = \widehat{Q}(S^{(\alpha)}) \cup Q$$

with equality if $* \in Q$. But now Lemma 11.2.4 gives

$$\widehat{Q}(S^{(\alpha)}) \cup Q \subseteq S^{(\alpha)*} \cup (S^{(\alpha)} \cap Q) \cup Q = S^{(\alpha+1)} \cup Q$$

with equality if $* \in Q$.

The leap to a limit ordinal is immediate. □

Consider $Q \in \mathcal{Q}S$ with $* \in Q$. Then, for each ordinal α we have

$$Q(\alpha) = S^{(\alpha)} \cup Q$$

so that by the time the $*$-sequence is stable the Q-sequence is also stable. To obtain a similar result when $* \notin Q$ we use the following.

Lemma 11.2.8. *Suppose $* \notin Q \in \mathcal{Q}S$. Then*

$$Q(1) \subseteq \{*\} \cup {\downarrow}Q \qquad Q(2) \subseteq {\downarrow}Q$$

and $* \notin Q(\infty)$.

Proof. For the first inclusion consider any $x \in \mathbb{S} - {\downarrow}Q$. We show that $x \notin Q(1)$. To this end let

$$U = \mathbb{S} - {\uparrow}x \qquad V = {\uparrow}x$$

so that both are open (by Examples 11.1.6 (b, a)). Also, if $q \in Q$ then $x \not\leq q \in \mathbb{S}$, so that $q \in U$. Thus $Q \subseteq U$. Finally, $U \cap V = \varnothing$ so that

$$Q(1) \subseteq U^- \subseteq V$$

and hence $x \notin U^-$ (since $x \notin V$) to give the required result.

For the second inclusion note that the set ${\downarrow}Q$ is closed (by Examples 11.1.6(d) and since Q is finite). Also $Q \subseteq \mathbb{S} \in \mathcal{O}S$ and then

$$\widehat{Q}(\{*\}) \subseteq (\{*\} \cap \mathbb{S})^- = \varnothing$$

so that

$$Q(2) = \widehat{Q}(Q(q)) \subseteq \widehat{Q}(\{*\} \cup {\downarrow}Q)) \subseteq \widehat{Q}(\{*\}) \cup \widehat{Q}({\downarrow}Q) \subseteq {\downarrow}Q$$

as required.

Finally, since $* \notin {\downarrow}Q$ we have $* \notin Q(2)$, and hence $* \notin Q(\infty)$. □

This brings us to our main result.

Theorem 11.2.9. *Let $\mathbb{S}$ be any tree with associated boss space S. Suppose $S^{(\theta)} = \{*\}$ for some ordinal θ. Then $Q(\theta) = Q$ for all $Q \in \mathcal{Q}S$.*

Proof. Consider any $Q \in \mathcal{Q}S$. By Lemma 11.2.7 we have

$$Q \subseteq Q(\theta) \subseteq \{*\} \cup Q$$

so that Q is one or other of these two extremes (since the extremes differ by at most one point).

Suppose $* \in Q$. Then the two extremes are the same, and we have the required equality.

Suppose $* \notin Q$. Then, by Lemma 11.2.8 we have $* \notin Q(\theta)$ and hence $Q(\theta) = Q$. □

Of course, this result does not say that every boss space is stacked, merely that its stacking property is determined entirely by the behaviour of the boss. Later we will see that some quite simple trees are not stacked. In contrast to this there is a simple property that ensures that the boss space of a tree is stacked.

Theorem 11.2.10. *Let $\mathbb{S}$ be a well-founded tree in which each branch is finite. Then its boss space S is T_1, sober, tightly packed and stacked, and hence tidy.*

Proof. By Theorem 11.1.9 it suffices to show that the boss space S is stacked. To this end let $X = S^{(\infty)}$, so that $X \in \mathcal{C}S$ with $X^* = X$ and hence $\odot(X) = \varnothing$. We require $X = \{*\}$. By way of contradiction suppose there is some $y \in X \cap \mathbb{S}$. This point lies on some branch of $\mathbb{S}$ which, by assumption, is finite. Thus by following this branch upwards we find some $x \in X$ with $I(x) \cap X = \varnothing$. But now $x \in \odot(X)$, which is the contradiction. □

We will use this result to produce examples of tidy spaces with arbitrarily large degrees of tidiness. To do that it is convenient to slightly reorganise the material.

As usual let $\mathbb{S}$ be a tree with boss space S. We will use certain closed subsets of S of the form $\{*\} \cup X$ for $X \subseteq \mathbb{S}$. We know that

$$\widehat{\{*\}}(\{*\} \cup X) = (\{*\} \cup X)^* = \{*\} \cup Y$$

for some $Y \subseteq X$. In this identity the boss $*$ does nothing very useful, it just sits there and watches. Thus we can more or less forget $*$ and work entirely in $\mathbb{S}$. To help with this we modify the notation of Definition 11.2.2.

Definition 11.2.11. Let $\mathbb{S}$ be a tree. For each $X \subseteq \mathbb{S}$ we use

$$x \in X^{\blacktriangledown} \iff x \in X \text{ and } I(x) \cap X \text{ is large}$$

(for $x \in \mathbb{S}$) to extract a subset of X. □

In this notation we have

$$(\{*\} \cup X)^* = \{*\} \cup X^{\blacktriangledown}$$

for each $X \subseteq \mathbb{S}$ where $\{*\} \cup X$ is closed. Thus we will use $(\cdot)^{\blacktriangledown}$ to do our calculations.

Consider a lower section X of $\mathbb{S}$. By Examples 11.1.6 (a) the complement $\mathbb{S} - X$ is open in the boss space, and hence $\{*\} \cup X$ is closed. We will use $(\cdot)^{\blacktriangledown}$ only on certain lower sections.

Definition 11.2.12. A lower section X of $\mathbb{S}$ is *bushy* if for each $x \in X$ either $I(x) \cap X$ is large or empty. □

For such a lower section X and $x \in X$, the set $I(x) \cap X$ is empty precisely when x is maximal in X. That is, when X is considered as a tree in its own right, the node is a leaf of X.

The following result is little more than a rephrasing of the definition of $(\cdot)^{\blacktriangledown}$.

Lemma 11.2.13. *For each bushy lower section X of $\mathbb{S}$, the nodes in $X^{\blacktriangledown}$ are precisely the non-leaves of X.*

We would like to iterate this pruning process $(\cdot)^{\blacktriangledown}$ to obtain a descending sequence of bushy lower sections. However, although for a bushy lower section X the result $X^{\blacktriangledown}$ is a lower section, it need not be bushy. Thus we will have to be a little careful with our use of $(\cdot)^{\blacktriangledown}$.

It's time to start looking at particular examples.

11.3 Full splitting trees

For our first examples we look at what must be the best known infinite trees.

Definition 11.3.1. Let I be an alphabet, and let $\mathbb{S}$ be the set of all words on I, including the empty word $\bot$. Thus each $x \in \mathbb{S}$ is a list (finite sequence)

$$x = \bot i_1 \dots i_l$$

of letters $i_1, \dots i_l \in I$. Here l is the length of x and $l = 0$ is allowed. These words are partially ordered by extension, thus $x \leq y$ (for words x, y) precisely when

$$y = x i_1 \dots i_l$$

for some sequence $i_1, \dots, i_l$ of letters (and again $l = 0$ is allowed). This makes $\mathbb{S}$ a well founded tree. □

For each word $x \in \mathbb{S}$

$$I(x) = \{xi \mid i \in I\}$$

is the set of immediate successors of x. The cardinality of $I(x)$ is important, and this is the cardinality of the parent alphabet I. There are several cases, some of which are pathological.

(0) When I is empty the tree $\mathbb{S}$ has a single node, the bottom $\bot$.

(1) When I is a singleton the tree $\mathbb{S}$ is a copy of the natural numbers.

(2) When I has just two members $\mathbb{S}$ is the Cantor tree, the full binary splitting tree.

(ω) When I is countably infinite $\mathbb{S}$ is the Baire tree.

(Ω) When I is uncountable the tree $\mathbb{S}$ is much more interesting.

Notice that although Theorem 11.1.9 does apply to this tree, Theorem 11.2.10 does not since $\mathbb{S}$ has infinite branches (except when I is empty).

The difference between a countable and an uncountable alphabet is dramatic.

Theorem 11.3.2. *For an alphabet I let $\mathbb{S}$ be the full I-splitting tree of Definition 11.3.1.*
(ω) If I is countable then $\mathbb{S}^{\blacktriangledown} = \varnothing$.
(Ω) If I is uncountable then $\mathbb{S}^{\blacktriangledown} = \mathbb{S}$.

Proof. For each node $x \in S$ we have $x \in \mathbb{S}^{\blacktriangledown}$ precisely when $I(x)$ is large, that is uncountable. This is never so for case (ω) and always so for case (Ω). □

This gives rise to some interesting properties.

Corollary 11.3.3. *If I is uncountable then the boss topology on $S = \{*\} \cup \mathbb{S}$ is not tidy.*

Proof. This follows directly from the previous theorem. Let Q be the compact saturated set $\{*\}$ so that

$$Q(1) = S^{(1)} = \{*\} \cup \mathbb{S}^{\blacktriangledown} = S$$

and so $Q(\alpha) = S$ for every ordinal α. Thus S is not tidy. □

Lemma 11.3.4. *If I is uncountable then the boss topology on $S = \{*\} \cup \mathbb{S}$ is not stacked.*

Proof. Consider $Q = \{*\}$. We have $\{*\} \ltimes S$ since

$$* \in U \Longrightarrow U^{-} = S$$

for every open set U. But

$$\{*\}^{-} = \{*\} \neq S$$

so S is not stacked. □

This has some implications for the spatiality of the patch assembly. By Lemma 8.5.11 if a space is not strongly stacked then there exists a compact saturated set Q and associated open filter F such that

$$v_F \neq [Q']$$

holds. We see that in this case,

$$v_F(\perp) = Q(\infty)' = \varnothing$$

whereas

$$[Q'](\perp) = Q'^{\circ} = \mathbb{S}$$

and so $\{*\}$ is one such Q. This leads to the following.

Theorem 11.3.5. *If I is uncountable then the boss topology on $S = \{*\} \cup \mathbb{S}$ has a patch assembly that is not spatial.*

Proof. First note that S is a T_1 space and as such has no wild points by Corollary 9.4.5 so $U_{\mathcal{POS}} = \pi$. We know that

$$\pi(v_F) = \pi([Q']) = Q'$$

but $v_F \neq [Q']$ and so π is not an isomorphism. This means that $U_{\mathcal{POS}}$ is not an isomorphism and thus $\mathcal{POS}$ is not spatial. □

For case (Ω) above there are some interesting subtrees.

The tree $\mathbb{S}$ is built up in layers $L_0, L_1, L_2, \ldots$ where

$$L_0 = \{\bot\} \qquad L_{r+1} = \{xi \mid x \in L_r, i \in I\}$$

for each $r < \omega$. In other words L_l is the set of words of length l, the set of all nodes

$$x = \bot i_1 \ldots i_l$$

for letters $i_1, \ldots i_l \in I$.

Now set

$$\mathbb{S}_0 = \varnothing \qquad \mathbb{S}_{r+1} = \mathbb{S}_r \cup L_r$$

(for $r < \omega$) to obtain a stratification

$$\mathbb{S}_0 \subseteq \mathbb{S}_1 \subseteq \ldots \subseteq \mathbb{S}_r \subseteq \ldots$$

of $\mathbb{S}$. Each $\mathbb{S}_r$ is a lower section of $\mathbb{S}$ and, for an uncountable alphabet I, each is bushy. Furthermore, the set of leaves of $\mathbb{S}_{r+1}$ is precisely the layer L_r. Thus Lemma 11.2.13 gives the following.

Lemma 11.3.6. *For an uncountable alphabet I we have*

$$\mathbb{S}_{r+1}{}^{\blacktriangledown} = \mathbb{S}_r$$

for each $r < \omega$.

Thus if we view $\mathbb{S}_{r+1}$ as a tree then r uses of $(\cdot)^{\blacktriangledown}$ will achieve $\mathbb{S}_0 = \{\bot\}$ and one more use will achieve $\varnothing$. We convert this tree into a boss space.

Theorem 11.3.7. *For each $r < \omega$ there is a space S_r which is T_1, sober, tightly packed and stacked with degree of tidiness exactly $r + 1$.*

This shows that the hierarchies of separation properties described in Sections 8.2 and 8.3 go at least as far as ω. The next job should be to produce a space of degree ω. However, it turns out that it is easier to deal with larger successor ordinals first. We return to ω later.

In Section 5.7 we indicated that even for a reasonably nice frame A there can be a block of NA with a complicated structure. Here we can give the details of that example.

The same observations were used again in Section 8.6, this time in connection with the V-points of a frame.

We can now fill in the missing details, but before that let's review what is needed.

Let A be a frame, let F be an open filter on A and consider the block

$$[v_F, w_F]$$

of NA, that is the block of all nuclei j on A with $\nabla(j) = F$. To show that this can be complicated we look at the principal lower section

$$I_F = [v_F(\bot), w_F(\bot)]_F$$

of the quotient A_F of A (obtained from the nucleus v_F). We want an example where this interval has many members.

We use the topology $A = \mathcal{O}S$ of the space $S = \{*\} \cup \mathbb{S}$ with the open filter given by $Q = \{*\}$. We have seen that

$$v_F(\varnothing) = S' = \varnothing \qquad w_F(\varnothing) = \{*\}'$$

so the problem is to exhibit a large number of closed sets $X \in \mathcal{C}S$ with $* \in X$ and $Q \ltimes X$. We obtain these from certain subtrees of $\mathbb{S}$. First observe that

$$\{*\} \ltimes X \iff X = X^*$$

by Lemma 8.5.8 and Corollary 11.2.5.

By definition, if $X \in \mathcal{C}S$ then

$$I(x) \cap X \text{ large} \Longrightarrow x \in X$$

for all $x \in \mathbb{S}$. In fact, we can strengthen this.

Lemma 11.3.8. *For each $X \in \mathcal{C}S$ we have*

$$I(x) \cap X \textit{ large} \Longrightarrow x \in X^*$$

for each $x \in \mathbb{S}$.

Proof. This is immediate from the definitions since $I(x) \cap X$ large means $x \notin \odot(X)$ so

$$x \in X - \odot(X) = X^*$$

as required. □

Theorem 11.3.9. *For each bushy subset $T \subseteq \mathbb{S}$ we have*

$$X = X^*$$

where $X = \{*\} \cup T$.

Proof. Since $* \in X$ we have $* \in X^*$. Consider any other $x \in X$. Then $x \in T$ and

$$I(x) \cap X = I(x) \cap T$$

and this is large by the choice of T. Thus $x \in X^*$ by Lemma 11.3.8. □

This gives us the closed sets we are looking for.

11.4 Some wide trees

To get beyond ω we need trees that are slightly more complicated than the set of all words on an alphabet. Where might we find such trees? To help with this let us say a tree is *standard* if it is well founded, rooted and each branch is finite. Thus Theorem 11.2.10 applies to standard trees. A standard tree is rooted in the sense that it has a bottom, there is a node $\bot$ for which $\bot \leq x$ for each node x. Let's look at a method by which standard trees can be grown.

The single node

$$\bot$$

is a standard tree, but not a very interesting one.

Given any indexed family

$$\mathbf{T} = (\mathbb{T}(j) \mid j \in J)$$

of standard trees we may form

$$\mathbb{S} = \frac{\mathbf{T}}{\bot}$$

by placing $\mathbf{T}$ above a new root $\bot$. This tree is standard because each branch of $\mathbb{S}$ is a branch of some $\mathbb{T}(j)$ with an extra node stuck onto the bottom end. In $\mathbb{S}$ the immediate successors of the root are the roots of the component trees $\mathbb{T}(j)$. In particular, we may have $\bot \notin \mathbb{S}^{\blacktriangledown}$ unless we make the index set J large. We will always ensure that.

Let I be some fixed uncountable set. Given a standard tree $\mathbb{T}$ we write $\mathbb{T} \cdot I$ for I copes of $\mathbb{T}$. We sit these copies above a new bottom to form a new tree

$$\mathbb{S} = \frac{\mathbb{T} \cdot I}{\bot}$$

in which $\bot \in \mathbb{S}^{\blacktriangledown}$. In fact

$$\mathbb{S}^{\blacktriangledown} = \frac{\mathbb{T}^{\blacktriangledown} \cdot I}{\bot}$$

where, of course, $\mathbb{T}^{\blacktriangledown}$ may be empty and hence $\mathbb{S}^{\blacktriangledown} = \{\bot\}$.

We refine and iterate this construction.

Given an indexed family

$$\mathbf{T} = (\mathbb{T}(j) \mid j \in J)$$

of standard trees we set

$$\mathbf{T} \cdot I = (\mathbb{T}(j) \cdot I \mid j \in J)$$

to obtain I copes of each component of $\mathbb{T}$. (It doesn't matter in which order the components are listed.)

Definition 11.4.1. A McTree is grown as follows.

- $\bot$ is a McTree.
- If $\mathbf{T}$ is a non-empty indexed family of McTrees then

$$\mathbb{S} = \frac{\mathbf{T} \cdot I}{\bot}$$

is a McTree.

There are no other McTrees. □

Observe that a McTree is a standard tree. In particular, each branch is finite. However, such a tree can be very wide, and we will exploit that feature.

Because the fixed indexing set I is uncountable, each McTree is bushy and so Lemma 11.2.13 gives the following.

Lemma 11.4.2. *For each McTree $\mathbb{S}$ the nodes in $\mathbb{S}^{\blacktriangledown}$ are precisely the non-leaves.*

In more pictorial detail we see that

$$\left(\frac{(\mathbb{T}(j)\cdot I \mid j\in J)}{\perp}\right)^{\blacktriangledown} = \frac{(\mathbb{T}(j)^{\blacktriangledown}\cdot I \mid j\in J)}{\perp}$$

where, of course, some of the $\mathbb{T}(j)^{\blacktriangledown}$ may be empty. From this we see that if $\mathbb{S}$ is a McTree then so is $\mathbb{S}^{\blacktriangledown}$, and hence we may iterate $(\cdot)^{\blacktriangledown}$ for a while to obtain less complicated McTrees.

Definition 11.4.3. For each McTree $\mathbb{S}$ let

$$(\mathbb{S}^{(\alpha)} \mid \alpha \in \mathsf{Ord})$$

be the ordinal indexed family of McTrees generated by

$$\mathbb{S}^{(0)} = \mathbb{S} \qquad \mathbb{S}^{(\alpha+1)} = \mathbb{S}^{(\alpha)\blacktriangledown} \qquad \mathbb{S}^{(\lambda)} = \bigcap\{\mathbb{S}^{(\alpha)} \mid \alpha < \lambda\}$$

for each sufficiently small ordinal α and sufficiently small limit ordinal λ. □

By Theorem 11.2.10 we know that if we iterate $(\cdot)^{\blacktriangledown}$ for too long then eventually we obtain $\varnothing$. This is why we use 'sufficiently small' ordinals in Definition 11.4.3. Returning to the picture we have

$$\left(\frac{(\mathbb{T}(j)\cdot I \mid j\in J)}{\perp}\right)^{(\alpha)} = \frac{(\mathbb{T}(j)^{(\alpha)}\cdot I \mid j\in J)}{\perp}$$

for each ordinal α that is not too large (and so at least one $\mathbb{T}(j)^{(\alpha)}$ is non-empty).

For each McTree $\mathbb{S}$ there is a least ordinal σ with $\mathbb{S}^{(\sigma)} = \varnothing$. Notice that this σ can never be a limit ordinal (for consider the role of the bottom of $\mathbb{S}$). What can σ be? To answer that we use a collection of particular McTrees.

Definition 11.4.4. Let

$$(\mathbb{R}(\alpha) \mid \alpha \in \mathsf{Ord})$$

be the ordinal indexed family of McTrees generated by

$$\mathbb{R}(0) = \perp \qquad \mathbb{R}(\alpha+1) = \frac{\mathbb{R}(\alpha)\cdot I}{\perp} \qquad \mathbb{R}(\lambda) = \frac{(\mathbb{R}(\alpha)\cdot I \mid \alpha < \lambda)}{\perp}$$

for each ordinal α and limit ordinal λ. □

Because of the limit clause in this construction these trees $\mathbb{R}(\cdot)$ can become very wide. Because of this, they can take a long time to cut down.

Lemma 11.4.5. *For each ordinal α we have $\mathbb{R}(\alpha)^{(\alpha)} = \perp$.*

Proof. We proceed by induction on α.

The base case, $\alpha = 0$, is immediate.

For the step case $\alpha \mapsto \alpha + 1$ the induction hypothesis gives

$$\mathbb{R}(\alpha)^{(\alpha)} = \perp$$

so that

$$\mathbb{R}(\alpha)^{(\alpha+1)} = 0$$

and hence

$$\mathbb{R}(\alpha+1)^{(\alpha+1)} = \frac{\mathbb{R}(\alpha)^{(\alpha+1)} \cdot I}{\bot} = \bot$$

as required.

For the leap to a limit ordinal λ we have

$$\mathbb{R}(\lambda)^{(\beta)} = \frac{(\mathbb{R}(\alpha)^{(\beta)} \cdot I \mid \alpha < \lambda)}{\bot}$$

for ordinals β that are not too large. Of course, some of the components $\mathbb{R}(\alpha)^{(\beta)}$ will be empty. In fact, by the induction hypothesis we have

$$\mathbb{R}(\alpha)^{(\beta)} = \varnothing$$

for $\alpha < \beta \leq \lambda$. The particular case $\beta = \lambda$ gives the required result. □

Of course, once we have pruned a tree down to its root, one more pruning will get rid of everything. Thus we have the following.

Lemma 11.4.6. *For each ordinal α the least ordinal σ with $\mathbb{R}(\alpha)^{(\sigma)} = \varnothing$ is $\alpha + 1$.*

Finally, by converting each tree $\mathbb{R}(\alpha)$ into its boss space we obtain the following extension of Theorem 11.3.7.

Theorem 11.4.7. *For each ordinal α there is a space S_α which is T_1, sober, tightly packed, and stacked with degree of tidiness exactly $\alpha + 1$.*

This shows that the hierarchies of Sections 8.2 and 8.3 are never ending in the sense that arbitrarily large degrees of tidiness can be achieved. What it doesn't show is that the degree of tidiness can be a limit ordinal. To do that we need a different construction.

11.5 The top down tree

This is our first, and only, example of a tree that is not wellfounded. We will see that, except for the top row, every node of this tree has uncountably many successors.

We begin as before by constructing a partial order. Take an uncountable set. For clarity we will use the real interval $(0, 1)$. We use the fact that an uncountable set can be split into uncountably many disjoint subsets, each of which is uncountable.

We need a function that will map $(0, 1)$ onto itself in such a way that each point in the interval is the image of uncountably many points. In other words, we require f such that

$$f\colon (0,1) \longrightarrow (0,1)$$

and for each $x \in (0, 1)$ the set $\{p \mid p \in (0, 1), f(p) = x\}$ is uncountable. For example, let f be the map that takes the even placed digits in the binary representation of p to form another real number between 0 and 1.

Definition 11.5.1. Let $\mathbb{U}$ be the set $(0,1) \times \mathbb{N}$ so that members of $\mathbb{U}$ are the pairs (p, n) where $p \in (0,1), n \in \mathbb{N}$. We define an ordering $\leq$ on $\mathbb{U}$ to be the smallest partial ordering satisfying the condition

$$(p, n) \geq (f(p), n+1)$$

so that for $n \geq 1$ each (p, n) has uncountably many points above it.

We refer to this as the *top down uncountably branching tree.* □

Unlike the infinite trees of Section 11.3, however, our tree has a top row but no bottom row. Although it is infinite, it does have leaves.

Definition 11.5.2. Let $S_\downarrow$ be $\mathbb{U} \cup \{*\}$ furnished with the boss topology. □

The reason this tree was constructed was to give an example of a space which is ω-tidy but not finitely tidy.

Lemma 11.5.3. *The space $S_\downarrow$ is ω-tidy but not n-tidy for any finite n.*

Proof. We define

$$\mathbb{U}_i = \{(p, n) \mid p \in (0,1), n \geq i\}$$

so that $\mathbb{U}_i$ is the tree $\mathbb{U}$ with the top i rows removed. We show by induction that

$$(S_\downarrow)^{(\alpha)} = \{*\} \cup \mathbb{U}_\alpha$$

for each finite ordinal α. Clearly the base case is true when $\alpha = 0$. Using the notation from Section 11.2 recall that

$$(\{*\} \cup \mathbb{U}_i)^* = \{*\} \cup \mathbb{U}_i{}^{\blacktriangledown}$$

and Lemma 11.2.13 tells us that $\mathbb{U}_i{}^{\blacktriangledown}$ is precisely the set of non leaves of $\mathbb{U}_i$, which is just $\mathbb{U}_{i+1}$. Thus the induction step follows by

$$S_\downarrow{}^{(\alpha+1)} = (S_\downarrow{}^{(\alpha)})^* = (\{*\} \cup \mathbb{U}_\alpha)^* = \{*\} \cup \mathbb{U}_i{}^{\blacktriangledown} = \{*\} \cup \mathbb{U}_{\alpha+1}$$

as required. This shows that $S_\downarrow$ is not n-tidy for any finite n.

When we make the jump to the limit ordinal ω we get

$$S_\downarrow{}^{(\omega)} = \bigcap\{S_\downarrow{}^{(\alpha)} \mid \alpha < \omega\} = \bigcap\{\{*\} \cup \mathbb{U}_i \mid i \in \mathbb{N}\} = \{*\}$$

which shows that $S_\downarrow$ is ω-tidy as claimed. □

We are still missing examples that have tidyness λ for limit ordinals λ that are greater than omega. I believe it should be possible to produce such examples, but we did not think it was worth spending the time to construct them.

Bibliography

[1] S. Abramsky and A. Jung: *Domain Theory*, Handbook of Logic in Computer Science, Volume 3, Clarendon Press, 1995.

[2] F. Borceux: *Handbook of categorical algebra, volume 2* (Cambridge University Press).

[3] B. Banaschewski and R. E. Hoffmann (editors) *Continuous Lattices* (Springer-Verlag 1981)

[4] M. H. Escardó: *The regular-locally-compact coreflection of a stably locally compact locale*, J. Pure and Appl. Alg. Vol. 157 Nr. 1 pp 41-55, 2001.

[5] R-E. Hoffmann (editor): *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere Nr. 27, Universität Bremen, 1982.

[6] K. H. Hofmann and J. D. Lawson: *On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom* pp 143-150 of [5].

[7] K.H. Hofmann et al.: *A Compendium of Continuous Lattices* (Springer-Verlag, 1980)

[8] K. H. Hofmann and M. W. Mislove: *Local compactness and continuous lattices* pp 209-248 of [3].

[9] P. T. Johnstone: *Stone Spaces* (Cambridge University Press).

[10] P. T. Johnstone: *The Vietoris Monad on the Category of Locales*, pp 162-179 of [5].

[11] P. J. Kirby: *A new look at counterexamples in topology*, MSc. thesis.

[12] M. W. Mislove: *Topology, Domain Theory and Theoretical Computer Science*, preprint.

[13] R. Sexton: *The use of the frame theoretic assembly as a unifying construct*, MSc. thesis.

[14] R. Sexton and H. Simmons: *Point-sensitive and point-free patch constructions*, submitted to J. Pure and Appl. Alg.

[15] H. Simmons: *The Vietoris modifications of a frame* http://www.cs.man.ac.uk/≈hsimmons

[16] M. B. Smyth: *Topology*, Handbook of Logic in Computer Science, Volume 2, Clarendon Press 1992.

[17] Lynn A. Steen and J. Arthur Seebach, Jr.: *Counterexamples in Topology*

[18] S. Vickers: *Topology via Logic* (Cambridge University Press, 1989)